MINIMIZING INTEGRALS IN CERTAIN CLASSES OF MONOTONE FUNCTIONS

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1. Introduction. This paper is concerned with the existence, uniqueness and representation of minimizing functions. It includes many results of [1] and [2]. Applications are discussed in [3].

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Extension to n variables and to more general integrands is of interest per se and is motivated by a variety of problems.

For example, let $\boldsymbol{x}$ ($\boldsymbol{y}$) be the random variable, maximum dilution (that is, unity minus concentration) of an insecticide I (J) which is lethal to an insect from a given population. Then

$$p(x, y) = \Pr\{\boldsymbol{x} > x \text{ or } \boldsymbol{y} > y\}$$

is the probability of death for an insect similtaneously dosed with respective dilutions x, y of I, J. Moreover

$$F(x, y) = 1 - p(x, y) = \Pr\{\boldsymbol{x} \leqq x \text{ and } \boldsymbol{y} \leqq y\}\ , \tag{1.1}$$

is the probability of survival and is a distribution function [5; pp. 78, 260]; hence $p(x, y)$ is nonincreasing in each variable and for each point-pair (x, y), (x', y'),

$$\Delta^2 p = p(x', y') - p(x', y) - p(x, y') + p(x, y) \leqq 0\ . \tag{1.2}$$

For each of selected pairs (x_i, y_j) let $\Delta\mu_{ij}$ insects be dosed and let α_{ij} denote the fraction of the sample which is killed. The maximum likelihood estimate $P(x, y)$ of $p(x, y)$ is that function, subject to the restrictions stated above, which maximizes the product

$$\prod p_{ij}^{\alpha_{ij}\Delta\mu_{ij}}(1 - p_{ij})^{(1-\alpha_{ij})\Delta\mu_{ij}}\ , \qquad p_{ij} = p(x_i, y_j)\ . \tag{1.3}$$

Equivalently, $P(x, y)$ minimizes the integral

$$-\int [\alpha \log p + (1-\alpha) \log (1-p)] d\mu\ , \tag{1.4}$$

in which μ describes the mass distribution consisting of masses $\Delta\mu_{ij}$ at

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