ON THE FUNCTIONAL REPRESENTATION OF CERTAIN ALGEBRAIC SYSTEMS

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1. Introduction, Definitions and Examples. In this paper an attempt is made to generalize the well-known representation theory of commutative Banach algebras by functions on the maximal ideals of the algebra [4]. The present paper is devoted almost exclusively to algebraic questions; topological aspects of the theory will be treated elsewhere.

In considering commutative algebras A over the complex field C, there are relatively few cases in which one can assert that the quotient A/M of the algebra by a maximal ideal is isomorphic to C. Apart from Banach algebras, there are the locally m-convex algebras of E. A. Michael [6] and R. Arens [1], and the 'algèbres à inverse continu' of L. Waelbroeck [8], [9] ($=Q$-algebras, in the terminology of Kaplansky [5], with continuous inversion). There are many interesting algebras which do not belong to either of these classes, and it would be desirable to have a theory to cover them as far as possible.

The basic idea is derived from the classical work of Carleman, von Neumann, and Stone on unbounded self-adjoint linear operators T in Hilbert space (see, for example, [7]). Here the analysis is carried out with the aid of the bounded transformations $(T-\lambda I)^{-1}$; the spectrum of T is the set of complex numbers λ such that $(T-\lambda I)^{-1}$ does not exist as a bounded transformation. This suggests that if we start with a commutative algebra A, and a suitable sub-algebra B (corresponding to the 'bounded' elements of A) we may be able to effect a useful analysis of A, and somehow represent an element $a \in A$ by a function whose values are those complex numbers λ such that $(a-\lambda e)^{-1}$ does not exists in B (e being the unit of A). It turns out that this is basically correct, although there are certain complications of detail. For instance, the representing functions may take infinite values; this is unavoidable. The space on which the functions are defined is that of the 'maximal B-ideals' or 'maximal ordinary B-ideals' of the algebra, not the space of maximal ideals in the ordinary sense.

Much of the theory of this paper applies to algebras over fields of fairly general type; for instance, many results are true for any algebraically closed field. It is no more difficult to develop the theory for the general case than for the case of the complex field. Let K be any

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