TWO THEOREMS ON TOPOLOGICAL LATTICES

ALEXANDER DONIPHAN WALLACE

A *topological lattice* is a pair of continuous functions

$$\wedge\colon L \times L \to L, \quad \wedge\colon L \times L \to L$$

(L a Hausdorff space) satisfying the usual conditions for lattice operations. A set A is *convex* if $x, y \in A$ and $x \leqq a \leqq y$ implies $a \in A$. This is equivalent to $A = (A \wedge L) \cap (A \vee L)$.

After proving a separation theorem involving a convex set we show that a compact connected topological lattice is a cyclic chain in the sense of G. T. Whyburn and that each cyclic element is a convex sublattice. In doing so we rely on some results recently obtained by L. W. Anderson.

THEOREM 1. *Let L be a connected topological lattice and let A be a convex set such that $L \setminus A$ is not connected. Then $L \setminus A$ is the union of the connected separated sets $(A \wedge L) \setminus A$ and $(A \vee L) \setminus A$ which are open (closed) if A is closed (open). If L is also compact then A is connected if it is either open or closed.*

Proof. Let $L \setminus A = U \cup V$ with $U^* \cap V = \phi = U \cap V^*$ and let $p \in U$, $q \in V$. The connected set $(p \wedge L) \cup (q \wedge L)$ meets both U and V; hence it meets A. Adjust the notation so that $(q \wedge L) \cap A \neq \phi$ and thus $q \in A \vee L$. If $(q \vee L) \cap A \neq \phi$ then $q \in A \wedge L$ and hence $q \in (A \wedge L) \cap (A \vee L) = A$. This being impossible we infer that $(q \vee L) \cap A = \phi$ and $q \in (A \vee L) \setminus A = (A \vee L) \setminus (A \wedge L)$. The connected set $(p \vee L) \cup (q \vee L)$ intersects U and V and so intersects A. But $(q \vee L) \cap A = \phi$ so that $(p \vee L) \cap A \neq \phi$ and hence $p \in A \wedge L$. Were $(p \wedge L) \cap A \neq \phi$ we would also have $p \in A \vee L$ and so $p \in A$, a contradiction. Thus $(p \wedge L) \cap A = \phi$ and $p \in (A \vee L) \setminus A = (A \vee L) \setminus (A \wedge L)$. Now take $y \in V$ and suppose that y is not in $A \vee L$ so that $(y \wedge L) \cap A = \phi$; then $(p \wedge L) \cap A \neq \phi$ since $(p \wedge L) \cup (y \wedge L)$ is a connected set meeting U and V. But this is contrary to the proven fact that $(p \wedge L) \cap A = \phi$. We conclude that $V \subset (A \vee L) \setminus A$ and, dually, that $U \subset (A \wedge L) \setminus A$. It follows that $L = (A \wedge L) \cup (A \vee L)$. Now $x \in (A \vee L) \setminus A$ and $x \in L \setminus V$ gives $x \in U \subset (A \wedge L) \setminus A$ and this contradicts the convexity of A. Hence $U = (A \wedge L) \setminus A$ and $V = (A \vee L) \setminus A$. To see that $U \wedge L = U$ we need only note that $x \in U$ gives $(x \wedge L) \cap A = \phi$ and thus $(x \wedge L) \cap V = \phi$ (since $x \wedge L$ is connected and contains x) and hence $x \wedge L \subset (A \wedge L) \setminus (A \vee L) = U$.

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