A THREE POINT CONVEXITY PROPERTY

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There exist an interesting variety of set properties determined by placing restrictions on each triple of points of the set. It is the purpose here to study those closed sets in the n-dimensional Euclidean space E_n (in particular the plane E_2) which satisfy the following condition.

DEFINITION 1. A set S in E_n is said to possess the *three-point convexity property* P_3 if for each triple of points x, y, z in S at least one of the closed segments xy, yz, xz is in S.

The principal result obtained in this paper appears in Theorem 2. In order to achieve this result a series of lemmas and theorems is first established. Most of these are also of independent interest.

1. Closed connected sets in E_n, $n \geqq 1$.

In this section we assume that S is a closed connected set in E_n, $n \geqq 1$. The concept of *local convexity* is a useful one for our purpose, so we restate the well-known definition.

DEFINITION 2. A set S is said to be *locally convex at a point* $q \in S$ if there exists an open sphere N with center at q such that $S \cdot N$ is convex. If a set is locally convex at each of its points, it is said to be *locally convex*.

NOTATION 1. The open segment determined by points x and y is denoted by (xy), whereas xy denotes the closed segment. The line determined by x and y is denoted by $L(x, y)$. The boundary of a set S is $B(S)$, and $H(S)$ denotes the closed convex hull of S. The symbol $+$ stands for set union, and the symbol $\cdot$ stands for set product.

THEOREM 1. *Let S be a closed connected set in E_n ($n \geqq 1$) which has property P_3. Then either S is convex or S is starlike with respect to each of its points of local nonconvexity (It may be starlike elsewhere).*

Proof. If S is locally convex, then by a theorem of Tietze [4, pp. 697–707], [2, pp. 448–449], the set S is convex, in which case it is starlike with respect to each of its points. Hence, suppose S is not locally convex, and let $q \in S$ be a point of local nonconvexity. This implies that in each spherical neighborhood N_i of q, there exist points

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