THE FREDHOLM EIGEN VALUES OF PLANE DOMAINS

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Introduction. The method of linear integral equations is an important tool in the theory of conformal mapping of plane simply-connected domains and in the boundary value problems of two-dimensional potential theory, in general. It yields a simple existence proof for solutions of such boundary value problems and leads to an effective construction of the required solution in terms of geometrically convergent Neumann-Liouville series. The convergence quality of these series is of considerable practical importance and has been discussed by various authors [4, 5, 6, 7]. It depends on the numerical value of the lowest nontrivial eigen value of the corresponding homogeneous integral equation which is an important functional of the boundary curve of the domain in question. Ahlfors [1, 10] gave an interesting estimate for this eigen value in terms of the extreme quasi-conformal mapping of the interior of this curve onto its exterior. Warschawski [15] gave a very useful estimate for it in terms of the corresponding eigen value of a nearby curve which allows often a good estimate of the desired eigen value in terms of a well-known one. This method is particularly valuable for special domains, for example, nearly-circular or convex ones.

It is the aim of the present paper to study the eigen functions and eigen values of the homogeneous Fredholm equation which is connected with the boundary value problem of two-dimensional potential theory. In particular, we want to obtain a sharp estimate for the lowest nontrivial eigen value in terms of function theoretic quantities connected with the curve considered. The steps of our investigation might become easier to understand by the following brief outline of our paper.

In § 1 we define the eigen values and eigen functions considered and transform the basic integral equation into a form which exhibits more clearly the interrelation with analytic function theory and extend the eigen functions as harmonic functions into the interior and the exterior of the curve. The boundary relations between these harmonic extensions are discussed and utilized to provide an example of a set of eigen functions and eigen values for the case of ellipses.

In § 2 we show the significance of the eigen value problem for the theory of the dielectric Green's function which depends on a positive parameter ε and is defined in the interior as well as the exterior of the curve. This Green's function has an immediate electrostatic interpretation and its theoretical value consists in the fact that it permits a

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