A TOPOLOGICAL CHARACTERIZATION OF SETS OF REAL NUMBERS

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We will say that a space E is of class L if E is a separable metric space which satisfies the following conditions:

(1) *Each component of E is a point or an arc (closed, open, or half-open), and no interior point of an arc-component A is a limit point of $E-A$.*

(2) *Each point of E has arbitrarily small neighborhoods whose boundaries are finite sets.*

The purpose of this note is to show that a necessary and sufficient condition that a space be homeomorphic to a set of real numbers is that it be of class L.

This gives an affirmative answer to a question raised by de Groot in [1].

In [2] L. W. Cohen proved that a separable metric space is homeomorphic to a set of real numbers if and only if it satisfies (1) above and (3) and (4) below:

(3) *E is zero-dimensional at each of its point-components.*

(4) *If p is an end point of an arc-component A, then the space $(E-A)\cup\{p\}$ is zero-dimensional at p.*

Any set of real numbers is clearly of class L. To prove the converse it is sufficient to show that every space of class L satisfies conditions (3) and (4). To this end it is clearly enough to show the following:

If X is a component of the space E of class L and ε is a positive number. there is a set $U(X, \varepsilon)$ which is both open and closed, contains X, and is contained in the union of X with the ε-neighborhoods of its endpoints (if any).

Suppose X is a component of a space E of class L and ε is a positive number. There exists an open set V which contains X but contains no point whose distance from X exceeds ε, such that the boundary B of V is finite; if X is a point, we can apply (2) directly to obtain V; if X is an arc, let V consist of X plus type (2) neighborhoods of the end points of X (if any).

Let G denote the sets of all points p of E such that E is the union of two mutually separated sets S_p and T_p, where S_p contains X and T_p contains p.

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