INDUCED HOMOLOGY HOMOMORPHISMS FOR SET-VALUED MAPS

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§ 1. **If X and Y are topological spaces,** a set-valued function $F: X \to Y$ assigns to each point x of X a closed nonempty subset $F(x)$ of Y. Let H denote Čech homology theory with coefficients in a field. If X and Y are compact metric spaces, we shall define for each such function F a vector space of homomorphisms from $H(X)$ to $H(Y)$ which deserve to be called the induced homomorphisms of F. Using this notion we prove two fixed point theorems of the Lefschetz type.

All spaces we deal with are assumed to be compact metric. Thus the group $H(X)$ can be based on a group $C(X)$ of projective chains [**4**]. Define the support of a coordinate c_i of $c \in C(X)$ to be the union of the closures of the kernels of the simplexes appearing in c_i. Then the intersection of the supports of the coordinates of c is defined to be the *support* $|c|$ of c.

If $F: X \to Y$ is a set-valued function, let $F^{-1}: Y \to X$ be the function such that $x \in F^{-1}(y)$ if and only if $y \in F(x)$. Then F is *upper* (*lower*) *semi-continuous* provided F^{-1} is *closed* (*open*). If both conditions hold, F is *continuous*. If $\varepsilon > 0$ is a real number, we shall also denote by $\varepsilon: X \to X$ the set-valued function such that $\varepsilon(x) = \{x' \mid d(x, x') \leqq \varepsilon\}$ for each $x \in X$.

Let A and B be chain groups with supports in X and Y respectively, and let $\varepsilon > 0$ be a number. A chain map $\varphi: A \to B$ is *accurate* with respect to a set-valued function $F: X \to Y$ provided $|\varphi(a)| \subset F(|a|)$ for each $a \in A$. Further, φ is *ε-accurate* with respect to F provided φ is accurate with respect to the composite function $\varepsilon F \varepsilon$.

(1) DEFINITION. A homomorphism $h: H(X) \to H(Y)$ is an *induced homomorphism* of a set-valued function $F: X \to Y$ provided that given $\varepsilon > 0$ there is a chain map $\varphi: C(X) \to C(Y)$ such that φ is ε-accurate with respect to F and $\varphi_* = h$.

We shall say that a homology homomorphism h is *nontrivial* provided the 0-dimensional component $h_0: H_0(X) \to H_0(Y)$ is not the zero homomorphism. It will appear that a continuous set-valued function need not have a nontrivial induced homomorphism.

The set of all induced homomorphisms of an arbitrary set-valued function is, under the usual operations, a vector space. If h_F and h_G are induced homomorphisms of upper semi-continuous functions $F: X \to Y$ and $G: Y \to Z$, then $h_G h_F$ is an induced homomorphism of GF. If F:

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