ON CERTAIN SUMS GENERATING THE DEDEKIND SUMS AND THEIR RECIPROCITY LAWS

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1. Introduction. Let $\{u\}=u-[u]$ denote the fractional part of u and let $((u))=\{u\}-\frac{1}{2}$. Dedekind sums are defined for example, by

$$s_1(h, k)=\sum_{\lambda=0}^{k-1}\left(\left(\frac{\lambda}{k}\right)\right)\left(\left(\frac{\lambda h}{k}\right)\right) \tag{1.1}$$

where h and k are relatively prime positive integers. These sums which were studied by Dedekind [7], and more recently by Rademacher and Whiteman [9], [12] in connection with the theory of the modular function $\eta(\tau)$, occur also in the theory of partitions and in a great number of special papers. (Cf. for example [1]-[13].) The most important property of $s_1(h, k)$ is the reciprocity law

$$s_1(h, k)+s_1(k, h)=(h^2+3hk+k^2+1)/(12hk)\ . \tag{1.2}$$

A few years ago, Apostol [1] (for $r=\nu$) and Carlitz [3] introduced and investigated the so-called generalized Dedekind sums

$$s_r^{(\nu)}(h, k)=\sum_{\lambda=0}^{k-1}P_{\nu+1-r}\left(\frac{\lambda}{k}\right)P_r\left(\frac{\lambda h}{k}\right) \qquad 0\leqq r\leqq \nu+1\ , \tag{1.3}$$

P_r denoting the well-known Bernoulli function defined by the expansion

$$ze^{uz}/(e^z-1)=\sum_{n=0}^{\infty}P_n(u)z^n/n! \qquad |z|<2\pi$$

for $0\leqq u<1$ and by $P_r(u)=P_r(\{u\})$ for u arbitrary real. They found the corresponding extensions of (1.2) too.

Now, we shall continue to develop these results in two directions. Next we give a systematic treatment of certain exponential sums (2.1), (2.3) generating

$$\mathfrak{S}_{m,n}\begin{pmatrix} a & b \\ & c \end{pmatrix}=\sum_{\nu=0}^{c-1}P_m\left(\frac{\lambda a}{c}\right)P_n\left(\frac{\lambda b}{c}\right) \qquad m, n=0, 1, 2, \cdots \tag{1.4}$$

with $(a, c)=(b, c)=1,\ c>0$. We obtain (among others) a three-term relation of new type (Theorem 1) which implies (in extended form) all the above reciprocity theorems (see (5.1)-(5.10)). Let us remark that the sum function (2.5) with other notations is also used in [6]. On the other hand, we get a functional equation for

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