NOTE ON NORMAL NUMBERS

CALVIN T. LONG

Introduction. Let α be a real number with fractional part $.a_1a_2a_3\cdots$ when written to base r. Let Y_n denote the block of the first n digits in this representation and let $N(d, Y_n)$ denote the number of occurrences of the digit d in Y_n. The number α is said to be *simply normal* to base r if

$$\lim_{n\to\infty} \frac{N(d,\ Y_n)}{n} = \frac{1}{r}$$

for each of the r distinct choices of d. α is said to be *normal* to base r if each of the numbers α, $r\alpha$, $r^2\alpha, \cdots$ are simply normal to each of the bases $r, r^2, r^3, \cdots$. These definitions, due to Emile Borel [1], were introduced in 1909. In 1940 S. S. Pillai [3] showed that a necessary and sufficient condition that α be normal to base r is that it be simply normal to each of the bases $r, r^2, r^3, \cdots$, thus considerably reducing the number of conditions needed to imply normality. The purpose of the present note is to show that α is normal to base r if and only if there exists a set of positive integers $m_1 < m_2 < m_3 < \cdots$ such that α is simply normal to base r^{m_i} for each $i \geqq 1$, and also to show that no finite set of m's will suffice.

Notation. We make use of the following additional conventions.

If B_k is any block of k digits to base r, $N(B_k, Y_n)$ will denote the number of occurrences of B_k in Y_n and $N_i(B_k, Y_n)$ will denote the number of occurrences of B_k starting in positions congruent to i modulo k in Y_n.

The term "relative frequency" will denote the asymptotic frequency with which an event occurs. For example, B_k occurs in (α), the fractional part of α, with relative frequency r^{-k} if $\lim_{n\to\infty} N(B_k, Y_n)/n = r^{-k}$.

Proof of the theorems. The following lemmas are easily proved.

LEMMA 1. *If* $\lim_{n\to\infty} \sum_{i=1}^{m} f_i(n) = 1$ *and if* $\liminf_{n\to\infty} f_i(n) \geqq 1/m$ *for* $i = 1, 2, \cdots, m$; *then* $\lim_{n\to\infty} f_i(n) = 1/m$ *for each* i.

LEMMA 2. *The real number* α *is simply normal to base* r^k *if and*

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