THREE TEST PROBLEMS IN OPERATOR THEORY

RICHARD V. KADISON AND I. M. SINGER

1. Introduction. In his tract [3] on infinite abelian groups, I. Kaplansky proposes three problems with which to test the adequacy of a purported structure theory for the subject. The problems are general with a certain intrinsic interest, and he comments there that they provide a worthy test in other subjects. In particular, Kaplansky has suggested these problems, suitably rephrased, in conversation as a test of a unitary equivalence theory for operators on a Hilbert space. In the order we treat them they are:

1. If A and B are operators acting on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$ and the operators $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ and $\begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$, acting in the obvious way on $\mathscr{H} \oplus \mathscr{H}$ and $\mathscr{K} \oplus \mathscr{K}$, are unitarily equivalent, is it true that A and B are unitarily equivalent?

2. If $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$ are unitarily equivalent is it true that B and C are unitarily equivalent?

3. If A and B are unitarily equivalent to direct summands of each other (that is, A equivalent to BF and B equivalent to AE, where E and F commute with A and B, respectively), are A and B unitarily equivalent?

A superficial examination provides examples which show that Problem 2 must, in general, be answered negatively. In fact infinite projections for B and C, one with an infinite and the other with a finite-dimensional orthogonal complement, and A an infinite-dimensional projection with an infinite-dimensional complement illustrates this. On the other hand, all three problems have an affirmative answer in the finite-dimensional case—Problem 3, trivially so, since E and F must be the identity operator on simple numerical-dimension grounds, and the other problems not at all trivially so (especially when approached from an elementary viewpoint).

Problem 3 has an affirmative answer, and a simple adaptation of the usual Cantor-Bernstein argument proves this. We shall give this problem no further attention except to note that it can be settled by

Received July 31, 1956. This work was supported by a contract with The Office of Naval Research.