ON SPACES WITH A MULTIPLICATION

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Introduction. This paper is divided into three parts, together with an appendix.

In the first part we discuss the homotopy theory of mappings into a space with a multiplication, such as a topological group. These spaces are more general than the group-like spaces considered by G. W. Whitehead in [6], and our treatment, as far as it goes, is quite different from his. In the second and third parts we apply the theory to the reduced product spaces of [2] and the loop-spaces of [4]. We arrive at useful new definitions of the Hopf construction and the Whitehead product, such that the relations between them are plainly exhibited. In many respects this completes the theory of the suspension triad as developed in [3].

Part I

Homotopy Theory of a Space with a Multiplication

1. Preliminary notions. Let S^r denote a topological r-sphere, with basepoint[1] e, where $r \geqq 1$. Let Z be a space with a basepoint, and let h: $S^p \times S^q \to Z$ be a map, where $p, q \geqq 1$. By the *sections* of h we means the maps f: $S^p \to Z$, g: $S^q \to Z$ which are defined by

$$f(x)=h(x, e)\ , \qquad g(y)=h(e, y) \qquad x \in S^p\ , \quad y \in S^q\ .$$

If h': $S^p \times S^q \to Z$ is another map with the same sections as h, then the two maps agree on the set of axes

$$\Sigma = S^p \times e \cup e \times S^q\ ,$$

and since the complement of Σ in $S^p \times S^q$ is an open $(p+q)$-cell the separation element $d(h, h') \in \pi_{p+q}(Z)$ is defined, as in § 10 below. Of course

$$d(h, h)=0\ . \tag{1.1}$$

In particular, let Z be a space with a multiplication; that is to say, there is a continuous product $x \cdot y \in Z$, where $x, y \in Z$, such that $x \cdot z^0 = x$ and $z^0 \cdot y = y$, where z^0 is the basepoint in Z. Let h be as before, and

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[1] When we consider a map, or homotopy, of one space into another it is always assumed that the image of the basepoint in the one is the basepoint in the other.