A THEOREM ON FLOWS IN NETWORKS

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1. Introduction. The theorem to be proved in this note is a generalization of a well-known combinatorial theorem of P. Hall, [**4**].

HALL'S THEOREM. *Let $S_1, S_2, \cdots, S_n$ be subsets of a set X. Then a necessary and sufficient condition that there exist distinct elements $x_1, \cdots, x_n$, such that $x_i \in S_i$ is that the union of every k sets from among the S_i contain at least k elements.*

The result has a simple interpretation in terms of transportation networks. A certain article is produced at a set X of origins, and is demanded at n destinations $y_1, \cdots, y_n$. Certain of the origins x are "connected" to certain of the destinations y making it possible to ship one article from x to y.

PROBLEM. *Under what conditions is it possible to ship articles to all the destinations y?*

An obvious reinterpretation of Hall's theorem shows that this is possible if and only if every k of the destinations are connected to at least k origins.

We shall now give a verbal statement of the generalization to be proved. A more formal statement will be given in the next section.

Let N be an arbitrary network or graph. To each node x of N corresponds a real number $d(x)$, where $|d(x)|$ is to be thought of as the demand for or the supply of some good at x according as $d(x)$ is positive or negative. To each edge (x, y) corresponds a nonnegative real number $c(x, y)$, the capacity of this edge, which assigns an upper bound to the possible flow from x to y.

The demands $d(x)$ are called *feasible* if there exists a flow in the network such that the flow along each edge is no greater than its capacity, and the net flow into (out of) each node is at least (at most) equal to the demand (supply) at that node.

An obviously necessary condition for the demands $d(x)$ to be feasible is the following.

For every collection S of nodes the sum of the demands at the nodes

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