BIORTHOGONAL SYSTEMS IN BANACH SPACES

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1. Introduction. We shall be interested, in this paper, in the following question: Given a biorthogonal system (x_n, f_n) in a separable Banach space B, under what conditions can one assert that the sequence $\{x_n\}$ constitutes a basis? The system (x_n, f_n) is called a biorthogonal system if

$$x_n \in B,\ f_n \in B^* \quad \text{and} \quad f_n(x_m) = \delta_{nm}.$$

We shall assume throughout the paper that $\|x_n\|=1$ and the sequence $\{x_n\}$ is fundamental. When the sequence $\{x_n\}$ constitutes a basis it will be called *regular* otherwise *irregular*.

2. Irregular systems. Let $\{x_n\}$ be an irregular sequence. (For example the trigonometric functions for $C(-\pi, \pi)$). The following definitions will be used.

$$\varphi_n(x) = \sum_{i=1}^{n} f_i(x) x_i$$

$$|||x||| = \sup \{ \|\varphi_n(x)\|,\ n=1, 2, 3, \cdots \}$$

Compare [4]

$$E_0 = \{x \mid \lim_{n\to\infty} \varphi_n(x) = x\}$$

$$E_1 = \{x \mid\ |||x||| < \infty\}$$

$$E_2 = \{x \mid \lim_{n\to\infty} \|\varphi_n(x)\| = \infty\}$$

$$E_3 = \{x \mid\ |||x||| = \infty\}.$$

We have $E_0 \subset E_1$ and $E_2 \subset E_3$. For regular systems $E_0 = E_1 = B$ and $E_2 = E_3 = \phi$ where ϕ is the null set. The system is regular if and only if the sequence $\{\|\varphi_n\|\}$ is bounded [2], and if the sequence $\{\|\varphi_n\|\}$ is not bounded the set

$$\bigcap_{n=1}^{\infty} \{x \mid\ \|\varphi_n(x)\| \leqq K\}$$

is nowhere dense [2], hence for irregular systems the set

$$E_1 = \bigcup_{k=1} \bigcap_{n=1} \{x \mid\ \|\varphi_n(x)\| \leqq K\}$$

is of the first category. Also $E_3 = B - E_1$ is dense and of the second

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