ASYMPTOTIC BEHAVIOR OF RESTRICTED EXTREMAL POLYNOMIALS AND OF THEIR ZEROS

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Introduction. Progress in the study of polynomials has recently been made in two directions: (i) asymptotic properties of sequences of polynomials of least norm on a given set (Leja, [7]; Davis and Pollak, [1]; Fekete, [3]; Walsh and Evans, [10]; Fekete and Walsh, [5]); (ii) geometry of the zeros of polynomials of prescribed degree minimizing a given norm on a given set, where one or more coefficients are preassigned (Zedek, [12]; Fekete, [4]; Walsh and Zedek, [11]; Fekete and Walsh, [6]). The object of the present paper is to combine these two trends, by studying the asymptotic properties of sequences of polynomials of least norm on a given set, where the polynomials are restricted by prescription of one or more coefficients.

If S is a given compact point set and $N[A_n(z), S]$ any norm on S of the polynomial $A_n(z) \equiv z^n + a_{1n}z^{n-1} + \cdots + a_{nn}$ we are interested in the asymptotic relations for (restricted) polynomials $A_n(z, N)$ of least N-norm

$$\lim_{n\to\infty} \nu_n^{1/n} = \tau(S) , \qquad \nu_n = N[A_n(z, N), S] , \tag{1}$$

$$\lim_{n\to\infty} |A_n(z, N)|^{1/n} = |\varphi(z)| , \tag{2}$$

where $\tau(S)$ is the transfinite diameter of S, $|\varphi(z)| \equiv e^{G(z)}\tau(S)$, $G(z)$ being Green's function with pole at infinity for the maximal infinite region K containing no point of S, and where (2) is considered uniformly on a more or less arbitrary compact set in K.

Part I is devoted primarily to (1); we show for instance that for the unit circle, with the first $k \equiv k(n)$ coefficients a_{jn} of the extremal polynomial $A_n(z, N)$ prescribed and uniformly of the order $O\left(\binom{n}{j}\right)$ in their totality, a necessary and sufficient condition for (1) for all such choices of coefficients is $k = o(n)$, where N is any classical norm. We prove similar results for other sets S. Part II is devoted primarily to (2); first we use as hypothesis the analogue of (1), namely

$$\{N[A_n(z), S]\}^{1/n} \to \tau(S) ,$$

for arbitrary polynomials $A_n(z)$; and then we use (1) as hypothesis, for extremal polynomials $A_n(z, N)$ with k prescribed coefficients and N

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