THE LATTICE OF INVARIANT SUBSPACES OF A COMPLETELY CONTINUOUS QUASI-NILPOTENT TRANSFORMATION

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An essential result in the study of a continuous linear transformation of a Banach space into itself is the specification of the lattice of proper closed subspaces of the Banach space which are invariant under the transformation. For certain classes of transformations the results which have been obtained in this direction may be regarded as complete, for example, for self-adjoint transformations in Hilbert space. The invariant subspaces for certain isometries in Hilbert space have been found by Beurling [2] whose results have been extended to unitary transformations by the author [3]. In general, however, little is known; in fact, it is not yet known that an arbitrary continuous linear transformation in Hilbert space has nontrivial closed invariant subspaces. A theorem of von Neumann guarantees that a completely continuous transformation in Hilbert space has such subspaces, while more recent work of Aronszajn and Smith [1] establishes the same result for any Banach space. For completely continuous transformations which contain only the point 0 in the spectrum (the quasi-nilpotent transformations), spectral theory can provide no information concerning the invariant subspaces, and the application of the result of Aronszajn and Smith only assures the existence of a nested sequence of closed invariant subspaces. Such a lattice of invariant subspaces is considerably simpler in structure than that usually encountered in spectral theory. It is the purpose of this note to show that more cannot be obtained, and that this very simple lattice does in fact occur. The three examples which follow illustrate this fact; the fourth example shows that not every completely continuous quasi-nilpotent transformation has such a lattice of invariant subspaces.

EXAMPLE 1. Let $\mathscr{H}$ be the Hilbert space consisting of all functions

$$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

analytic in $|z|<1$ with Taylor coefficients in l^2:

$$\sum_{n=0}^{\infty} |a_n|^2=\|f\|^2<\infty .$$

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