ON GENERALIZED EUCLIDEAN AND NON-EUCLIDEAN SPACES

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Introduction. The present paper develops necessary and sufficient conditions that a complete, convex, metric space with extendible segments shall be generalized euclidean, r-hyperbolic, r-spherical, or r-elliptic. Blumenthal and others have given four-point conditions which characterize these generalized spaces among certain classes of spaces, and the results of this paper follow the general plan of these earlier works.

1. Definitions, notation and previous results. Unless otherwise noted all terms used have the same meanings as those given in [1]. The distance between two points p and q of a semi-metric space is denoted by pq, a point s distinct from p and from q is *between* p *and* q, denoted by psq, provided $ps+sq=pq$, and a triple of points (not necessarily distinct) is a *mid-point triple*, denoted by (psq), provided $ps=sq=pq/2$. A metric space is said to be generalized {euclidean, r-hyperbolic, r-spherical, r-elliptic} provided each of its n-dimensional subspaces is congruent with $\{E_n, H_{n,r}, S_{n,r}, \mathscr{E}_{n,r}\}$, where these four symbols represent n-dimensional euclidean, hyperbolic, spherical, elliptic space respectively, the last three of space constant $r>0$. A metric space is said to have the weak {euclidean, r-hyperbolic, r-spherical, r-elliptic} four-point property provided each of its quadruples containing a triple of points congruent to three points of $\{E_1, H_{1,r}, S_{1,r}, \mathscr{E}_{1,r}\}$ is itself congruent to four points of $\{E_2, H_{2,r}, S_{2,r}, \mathscr{E}_{2,r}\}$. A space has the feeble {euclidean, r-hyperbolic, r-spherical, r-elliptic} four-point property provided each quadruple containing a mid-point triple is congruently imbeddable in $\{E_2, H_{2,r}, S_{2,r}, \mathscr{E}_{2,r}\}$. The weak property obviously implies the feeble property.

THEOREM 1 (Blumenthal [2]). *A complete, convex, externally convex metric space is generalized euclidean if and only if it has the feeble euclidean four-point property.*

Defining a conjugate space as one with finite metric diameter $\delta>0$ and having the further property that corresponding to each pair of points p, q of the space with $0<pq<\delta$ there exist points p^*, q^* of the space with pqp^*, qpq^*, and $pp^*=qq^*=\delta$ all holding, Hankins [4] has shown the following.

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