INTERIOR VARIATIONS AND SOME EXTREMAL PROBLEMS FOR CERTAIN CLASSES OF UNIVALENT FUNCTIONS

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1. Introduction. The theory of regular univalent functions in the unit circle U_z has been developed for various subclasses, for example, the class of real univalent functions which leads to symmetric domains, the class of bounded univalent functions whose image domain lies with in the unit circle and the functions for which the image domains are convex or star-like. The approach through the calculus of variations has been used very successfully towards the solution of extremal problems belonging to the various classes and also towards the determination of the extremal domains. The purpose of the present paper is to show how the method of interior variations due to Schiffer [1] can be adapted for the following subclasses:

(i) The class V of symmetric regular univalent-functions $f(z)$ in U_z which have the form $f(z)=z+\sum\limits_{n=2}^{\infty} a_n z^n$ with real a_n. In particular we show that if $\varphi(a_2, a_3, \cdots, a_n; \bar{a}_2, \bar{a}_3, \cdots, \bar{a}_n)$ is a real valued function which is symmetric and analytic in a_ν and $\bar{a}_\nu$ ($\nu=2, 3, \cdots, n$) and where $\{a_n\}$ are the coefficients in the power series expansion of the more general class V_1 of regular univalent functions then, under the assumption that the function $f(z)$ whose coefficients $\{a_\nu\}$ maximize $\varphi(a_2, \cdots, a_n, \bar{a}_2, \cdots, \bar{a}_n)$ is symmetric, the functional differential equation satisfied by $f(z)$ in the general class V_1 is the same as the functional differential equation satisfied by $f(z)$ in the class V.

(ii) The class S of bounded univalent functions $f(z)$ in U_z which are normalized so that $f(0)=0$, $|f(z)| \leq 1$ and at a fixed point $\zeta \in U_z$, $f(\zeta)=\omega$. In particular we find the functions which maximize or minimize $|f'(\zeta)|$.

(iii) The class Σ of bounded univalent functions $f(z)$ in U_z which are real on the real axis and are normalized so that $f(0)=0$, $|f(z)| \leq 1$ and at a fixed point ζ on the real axis $f(\zeta)=\omega$. In particular we find the functions which maximize or minimize $f(\eta)$ for real $\eta \in U_z$.

We observe that the existence and uniqueness of the solutions of these problems is assured because the families of functions belonging to the classes V, S and Σ are normal and compact.

2. Real univalent functions. Let D be the image in the W-plane