CONVEXITY OF ORLICZ SPACES

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In a paper [1] which appeared in 1936, J. A. Clarkson defined a property of Banach spaces known as uniform convexity. Let $\|f\|$ denote the norm of an element f of such a space and let $\{f'_n, f''_n\}$ be any sequence of pairs of elements such that $\|f'_n\|=\|f''_n\|=1$ and $\lim_{n\to\infty} \frac{1}{2}\|f'_n+f''_n\|=1$. The space is said to be *uniformly convex* if these conditions imply that $\lim_{n\to\infty}\|f'_n-f''_n\|=0$. It has been shown [2] that an equivalent definition is one in which the condition $\|f'_n\|=\|f''_n\|=1$ may be replaced with the weaker $\|f'_n\|\leqq 1$ and $\|f''_n\|\leqq 1$. Clarkson has been successful in showing that the Lebesgue spaces L_p are uniformly convex if $p\neq 1$ and that L_1 is not uniformly convex. The convexity properties of more general classes of Banach spaces have been investigated by M. M. Day [3], I. Halperin [4] and E. J. McShane [7].

A concept of convexity related to uniform convexity has been described and is termed *strict convexity*. It is defined in the following manner. Let f', f'' be any pair of elements in a Banach space such that $\|f'\|=\|f''\|=1$ and $\frac{1}{2}\|f'_n+f''_n\|=1$. The space is said to be strictly convex if these conditions imply that $\|f'-f''\|=0$. In a Euclidean space, strict convexity corresponds geometrically to the property that the unit sphere $\|f\|=1$ does not contain a segment. We remark that, if a space has the property of uniform convexity, then it possesses that of strict convexity as well; however, the converse implication is generally untrue.

The principal objective of this paper is to investigate the conditions which an Orlicz space [9] must satisfy to be uniformly convex. Also the related problem of determining the conditions for strict convexity is considered. A solution to both of these questions has been presented which may be regarded as complete in the sense that both the necessary and sufficient criteria are developed.

We begin by formulating the definitions of Orlicz spaces in accordance with the notations to be used subsequently. Except in minor details we shall adopt the standard conventions. Let $v=\varphi(u)$ be a monotonically nondecreasing function not identically zero, defined for all $0\leqq u$ such that $\varphi(u)=\varphi(u-)$ and $\varphi(0)=0$; also, let $\overline{\varphi}(u)$ denote the associated function $\overline{\varphi}(u)=\varphi(u+)$. Let $u=\psi(v)$ be the function inverse to $\varphi(u)$ which is defined by the relations:

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