THE COEFFICIENT REGIONS OF STARLIKE FUNCTIONS

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1. The coefficient regions of schlicht functions have been studied at some length by Schaeffer, Schiffer, and Spencer [2, 3]. Properties of these coefficient regions are obtained only with difficulty, and in particular the actual coefficient regions can be computed only with a great deal of labor [2]. In fact, the computations necessary to determine the coefficient region of (a_2, a_3, a_4) probably would be prohibitive.

The class of starlike functions is of course much simpler in behavior. Since $f(z)=z+a_2z^2+a_3z^3+\cdots$ is starlike if and only if $zf'(z)/f(z)$ has a positive real part in $|z|<1$, one might say that everything is known about such functions. However, in practice, our rather complete knowledge about functions with positive real part proves difficult to apply back to the class of starlike functions. This is easily seen to be true by noting the number of papers on starlike functions which appear every year.

In an earlier paper, the writer presented a new variational method in the class of starlike functions. It is the purpose of this paper to apply this variational method to find the coefficient regions for starlike functions.

Let S^* be the class of all normalized functions $f(z)=z+a_2z^2+a_3z^3+\cdots$, schlicht and starlike in the unit circle. Let $\boldsymbol{V}_n^*$ be the $(2n-2)$ dimensional region composed of all points $(a_2, a_3, \cdots, a_n)$ belonging to the functions of S^*. Since the class of functions $p(z)$ with $p(0)=1$, regular and having a positive real part in $|z|<1$, is a compact family, so is S^*. Thus $\boldsymbol{V}_n^*$ is a closed domain (i.e., the closure of a domain).

We will study $\boldsymbol{V}_n^*$ by determining its cross sections with $a_2, a_3, \cdots, a_{n-1}$ held fixed. In § 2, a simple proof of the fact that each such cross section is convex is given. It is then shown that any point on the boundary of this cross section must lie on a particular circle, and thus that the cross section itself is a circle. The actual equations for the region $\boldsymbol{V}_n^*$ can be determined for each n by means of a simple recursion, but the calculation becomes tedious after the first few n.

2. For fixed $a_2, a_3, \cdots, a_{n-1}$, let $\boldsymbol{C}_n^*=\boldsymbol{C}_n^*(a_2, \cdots, a_{n-1})$ be the two dimensional cross section of $\boldsymbol{V}_n^*$ in which a_n varies.

LEMMA 1. *$\boldsymbol{C}_n^*$ is a closed, convex set.*

Received June 24, 1957. The work reported on here was done while the writer held a National Science Foundation post doctoral Fellowship. The writer wishes to thank Professor M. Schiffer for many helpful conversations during the course of this work.