ADDITION THEOREMS FOR SOLUTIONS OF THE WAVE EQUATION IN PARABOLIC COORDINATES

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1. Introduction. The wave equation

$$\Delta U + k^2 U = 0$$

admits solutions of the form

$$U_{\kappa,\mu} = A_{\kappa,\mu}(\xi) B_{\kappa,\mu}(\eta) C_{\kappa,\mu}(\phi)$$

if the coordinate system is such that separation of variables is possible. ξ, η and ϕ are the three independent variables, and κ and μ represent arbitrary complex parameters. In general $U_{\kappa,\mu}$ will not be regular and one-valued over the whole space, but will be so for special values of κ and μ. Let ξ', η' and ϕ' be functions of ξ, η, and ϕ resulting from a translation or rotation of the coordinate system; then a relation which expresses $U_{\kappa,\mu}(\xi', \eta', \phi')$ as a summation of terms of the form $U_{\kappa,\mu}(\xi, \eta, \phi)$ is called an addition theorem.

Addition theorems for cylindrical and spherical coordinate systems are well known. These are the addition theorems for Bessel and Hankel functions, Legendre polynomials, spherical harmonics, Mathieu functions and spheroidal wave functions (see Meixner and Schäfke [5] and Erdélyi [2]).

It is proposed to derive such addition theorems for those functions of the paraboloid of revolution which are regular and one-valued in the whole space. As will be seen subsequently, these restrictions are not always necessary. That such theorems might exist can be inferred from the invariance of ΔU under rotations and translations of space, and from the fact that the family of solutions that are everywhere regular and one-valued will be mapped onto itself by motions of space.

It is possible to derive several of these theorems by using known addition theorems. For example, it is possible to derive linear relations between the functions of the paraboloid of revolution and spherical harmonics. Since an addition theorem under a rotation of coordinates is known for the latter functions, it is possible to derive one for the functions of the paraboloid of revolution.

2. The functions of the paraboloid of revolution. The introduction

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