ON LINEAR SYSTEMS WITH INTEGRAL VALUED SOLUTIONS

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1. Introduction. We consider a system of linear equations and inequalities in k variables

$$(1.1) \qquad Ax=b\,, \quad x \geqq o\,,$$

where the matrix A has r rows, k columns, and rank less than k.

Assuming the system consistent, the solution set is a convex polyhedron P in k-space. A solution x^0 that satisfies k independent relations of (1.1) as equations, is a vertex of P, and conversely. Such solution is generally called basic or extremal, and is equivalently defined by the property, that the columns of A corresponding to nonzero coordinates of x^0 are independent. Basic solutions are of particular interest in problems where a linear functional is extremised over P, the extremum then being assumed at a vertex or at all points of a positive dimensional face F of P, that is, the convex hull of the vertices of F. In such problems the interest is often restricted to the integral valued basic solutions as the only ones that have meaning in the application. Now given P, any vertex of P can appear as solution of an extremum problem for some linear functional, and a question of interest is: when, that is for which systems (1.1), are all the vertices of P integral valued.

Directing the attention to the system

$$(1.2) \qquad Ax=b\,,$$

we may, slightly generalizing, respectively specializing, carry over the definition and the question:

(1.3) DEFINITION. A solution x^0 of (1.2) is *basic*, when its nonzero coordinates correspond to linearly independent columns of A.

(1.4) QUESTION. Which systems (1.2) have all their basic solutions integral valued?

Obviously (1.4) is not equivalent to the same question for systems (1.1); the basic solutions of (1.2) contain those of (1.1); but they may also contain others, namely such with negative integral coordinates. Hence (1.4) asks more and will therefore yield a smaller family of

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