THE FIVE-POINT DIFFERENCE EQUATION WITH PERIODIC COEFFICIENTS

TOMLINSON FORT

The five-point difference equation described in § 1 has most of the important second order partial difference equations as special cases and as limiting forms of these the more important partial differential equations of the second order. In the present paper all coefficients are assumed periodic in the same one of the two independent variables. The purpose of the paper is the study of the form of the general solution as affected by the periodic character of the coefficients. This study centers around the roots of the characteristic equation and so-called semi-periodic solutions. The reader is referred to the theorem of § 5 for a precise statement of results.

1. General discussion. Let us be given the five-point equation

$$(1)\qquad k_1(i,j)y(i-1,j)+k_2(i,j)y(i+1,j)+k_3(i,j)y(i,j-1)+k_4(i,j)y(i,j+1)+k_5(i,j)y(i,j)=0$$

where k_1, k_2, k_3, k_4 and k_5 are defined for integral values of i and j over the rectangle $1\leq i\leq n\omega-1$, $1\leq j\leq \omega-1$ where $n>1$ and $\omega>1$ are integers. This rectangle will be called the *defining rectangle* and will be denoted by R. We assume moreover that

$$(2)\qquad k_\nu(i+\omega,j)=k_\nu(i,j)\ ,\qquad \nu=1,2,3,4,5$$

and that neither, k_1, k_2, k_3, nor k_4 is zero at any point of R.

A *solution* of (1) is a function of (i,j) defined at points of R and at the border points $(i=0,\ j=1,2,\cdots,\omega-1)$, $(i=n\omega,\ j=1,2,\cdots,\omega-1)$, $(j=0,\ i=1,2,\cdots,n\omega-1)$, $(j=\omega.\ i=1,2,\cdots,n\omega-1)$ and which satisfies (1) at all points of R. Notice that this second set of points, namely R plus the border points, form a lattice which is rectangular except that its corner points are missing. It will be referred to as the rectangle S.

A *fundamental domain* is a set of points of S such that there exists one and only one solution taking on prescribed arbitrary values at each point of the set.

All fundamental domains[1] contain the same number of points. We denote this number by L. For the rectangle S

Received December 6, 1956. This paper was sponsored by the Office of Ordnance Research of the U. S. Army.

[1] For a detailed discussion see T. Fort, Amer. Math. Monthly, **62**, (1955), 161.