ON THE CASIMIR OPERATOR

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The Casimir operator is an important tool in the study of associative [4], Lie [4] and alternative algebras [7]. However its use has been for algebras of characteristic 0. We give a new definition of the Casimir operator for associative, Lie and alternative algebras, which keeps desirable properties of the usual Casimir operator and which is useful for arbitrary characteristic.

We show that under certain conditions our Casimir operator is the identity transformation and for non-degenerate alternative (or associative) algebras we show that it is the transformation into which the identity element of the algebra maps. We apply our results to obtain the first Whitehead lemma for non-degenerate alternative algebras of arbitrary characteristic. We also obtain a special case of the Levi theorem for Lie algebras of prime characteristic.

1. The Casimir Operator. Let $\mathfrak{A}$ be an associative, Lie or alternative algebra with basis $e_1, e_2, \cdots, e_n$ over an arbitrary field $\mathfrak{F}$. For uniformity we use the notation $x \rightarrow S_x$ for a representation of $\mathfrak{A}$, where if $\mathfrak{A}$ is alternative we mean the S_x part of a representation $x \rightarrow (S_x, T_x)$. If $\mathfrak{A}$ is a Lie or associative algebra, $f(x, y)=t(S_xS_y)$ where t is the trace function, is an invariant symmetric bilinear form. In [7, p. 444] it is shown that if $\mathfrak{A}$ is alternative this form is invariant if $\mathfrak{F}$ is not of characteristic 2. For arbitrary characteristic we have

$$\begin{aligned} t(S_xS_{yz}) &= t(S_xS_yS_z+S_xT_yS_z-S_xS_zT_y) \\ &= t(S_xS_yS_z+S_xT_yS_z-T_yS_xS_z)=t(S_{xy}S_z) \ . \end{aligned}$$

Similarly $t(T_xT_y)$ is invariant.

We call $\mathfrak{A}$ *non-degenerate* if $t(R_xR_y)$ is non-degenerate where R is the representation of right multiplications. It can be shown that this is equivalent to the non-degeneracy of the bilinear form $t(L_xL_y)$ of the left multiplications. It is well known that if $\mathfrak{A}$ is a non-degenerate alternative (or associative) algebra it is a direct sum of simple algebras. Dieudonne [3] has shown that this is also true for Lie algebras.

If $\mathfrak{A}$ is semi-simple and $\mathfrak{F}$ is of characteristic 0, the usual Casimir operator Γ_S^* for the representation S is defined as follows: Let $\mathfrak{N}$ be the set of all x of $\mathfrak{A}$ such that $t(S_xS_y)=0$ for all y of $\mathfrak{A}$. Then $\mathfrak{A}=\mathfrak{N}\oplus\mathfrak{C}$ where $\mathfrak{N}$ and $\mathfrak{C}$ are semi-simple ideals of $\mathfrak{A}$. Let $e_1', e_2', \cdots, e_k'$ be the

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