AN ULTRASPHERICAL GENERATING FUNCTION

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1. **Introduction.** Let $P_n^{(\alpha,\alpha)}(v)$ denote ultraspherical polynomials and let

$$(1)\qquad \begin{aligned} w&=2(v-t)(1-2vt+t^2)^{-1/2}\,,\\ g&=1-2vt+t^2\,,\\ y&=-tu(1-2vt+t^2)^{-1/2}\,,\\ r&=(1-2yw+y^2)^{1/2}\,,\end{aligned}$$

with the roots to be those assuming the value 1 for $t=0$. Then this note will prove that

$$(2)\qquad g^{-\alpha-1/2}\,{}_2F_1\left[\begin{matrix}c,\ 1+2\alpha-c; & \dfrac{1-y-r}{2}\\ 1+\alpha\quad ; & \end{matrix}\right]{}_2F_1\left[\begin{matrix}c,\ 1+2\alpha-c; & \dfrac{1+y-r}{2}\\ 1+\alpha\quad ; & \end{matrix}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_3F_2\left[\begin{matrix}-n,\ c,\ 1+2\alpha-c; & u\\ 1+\alpha,\ 1+2\alpha\quad ; & \end{matrix}\right]P_n^{(\alpha,\alpha)}(v)t^n\,,$$

valid for t sufficiently small. In (2), c is an arbitrary parameter. Equation (2) is a direct generalization of Rice's result given in [8, equ. 2.14], to which it reduces for $\alpha=0$. (A different generalization of Rice's result is given in [3].) For c the non-positive integer $-k$, the left side of (2) reduces to a product of ultraspherical polynomials:

$$(3)\qquad g^{-\alpha-1/2}\frac{k!k!}{(1+\alpha)_k(1+\alpha)_k}P_k^{(\alpha,\alpha)}(r+y)P_k^{(\alpha,\alpha)}(r-y)$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_3F_2\left[\begin{matrix}-n,\ -k,\ 1+2\alpha+k; & u\\ 1+\alpha,\ 1+2\alpha\quad ; & \end{matrix}\right]P_n^{(\alpha,\alpha)}(v)t^n\,.$$

In addition, this note will show other results on ultraspherical polynomials. Further, it will provide a new way of deriving some results of Weisner. These will be shown later.

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2. **A preliminary result.** It will be established in this section that

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