ON THE CONSTRUCTION OF R-MODULES AND RINGS WITH POLYNOMIAL MULTIPLICATION

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1. Introduction. Let R be a ring and let R^+ be the additive group of R. If $R^+ = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ is a direct sum of subgroups S_i, then each element of R can be written as an n-tuple $(s_1, s_2, \cdots, s_n)$, $s_i \in S_i$, $i = 1, 2, \cdots, n$, and multiplication in R is given by n mappings

$$f_k : S_1 \times S_2 \times \cdots \times S_n \times S_1 \times S_2 \times \cdots \times S_n \to R^+ , \qquad k = 1, 2, \cdots, n ,$$

where $f_k(s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_n)$ is the k-th component of the product $(s_1, s_2, \cdots, s_n) \cdot (t_1, t_2, \cdots, t_n)$. The distributive laws in R imply that the mappings f_k are additive in the first n and in the last n arguments. If $S_1, S_2, \cdots, S_n$ are ideals in R, then

$$f_k(s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_n) = s_k t_k , \qquad k = 1, 2, \cdots, n ,$$

which is a homogeneous quadratic polynomial with integral coefficients in the arguments.

If R is a commutative ring with identity, and if M is a free (left) R-module with basis $e_1, e_2, \cdots, e_n$, then M is an algebra over R if and only if there exist elements $\gamma_{ijk} \in R$ such that multiplication in M is defined by

$$\left(\sum_{i=1}^{n} s_i e_i \right) \cdot \left(\sum_{j=1}^{n} t_j e_j \right) = \sum_{i,j,k=1}^{n} \gamma_{ijk} s_i t_j e_k .$$

The k-th coordinate of the product,

$$f_k(s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_n) = \sum_{i,j=1}^{n} \gamma_{ijk} s_i t_j ,$$

is a mapping

$$f_k : \overbrace{R^+ \times R^+ \times \cdots \times R^+}^{2n} \to R^+$$

which is additive in the first n and last n arguments, and which is a homogeneous quadratic polynomial with coefficients in R in the arguments.

These examples suggest the investigation of polynomial mappings with the indicated additive properties, and a discussion of the problem of constructing R-modules and rings which have an additive group which is the direct sum of ideals of a ring R, and for which the multiplication

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