MULTIPLICATIVE NORMS FOR METRIC RINGS

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1. Introduction. In his paper [19], S. Mazur stated two results concerning real normed algebras. The first of these, which asserted that the only normed division algebras over the real field were the real field, the complex field, and the division ring of real quaternions, was essentially proved by Gelfand in [10] and by Lorch in [17]. Elementary proofs of that result have also been given by Kametani [13] and Tornheim [26], while generalizations in various directions have been given by Kaplansky [16], Arens [4] and Ramaswami [23].

The second of the results given by Mazur was that a real normed algebra such that $||xy||=||x||\,||y||$ for all x and y mnst again be isomorphic to the real field, the complex field, or the division ring of real quaternions. This result was generalized in [8] by R.E. Edwards, who showed that the same conclusion holds for a Banach algebra under the weaker hypothesis that $||x||\,||x^{-1}||=1$ for all elements x which have inverses x^{-1}. A. A. Albert has also obtained results in [1], [2] and [3] similar to the second of Mazur's results.

In this paper, the second result of Mazur is generalized for certain types of metric rings. It is shown in section 6 that such rings must be division rings if the condition $||xy||=||x||\,||y||$ for all x and y holds. Similar results hold under the weaker assumption that $||x||\,||x^{-1}||=1$ for every element x which has an inverse x^{-1}. Under suitable additional conditions on the metric rings under discussion, it is shown in § 7 that the results just mentioned may be strengthened to assert that the ring is not only a division ring but is isomorphic to the real field, the complex field, or the division ring of real quaternions. Finally, the results on metric rings are applied to real normed algebras to obtain the results of Mazur and Edwards under weaker assumptions.

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2. Topological rings, metric rings, regular and singular elements. We shall first introduce some pertinent definitions and recall some elementary results concerning topological rings and metric rings. By a topological ring is meant a structure R which is at once a Hausdorff

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