MONOTONE COMPLETENESS OF NORMED SEMI-ORDERED LINEAR SPACES

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Introduction. Let R be a continuous semi-ordered linear space, namely, a semi-ordered linear space where, for any sequence $x_\nu \geqq 0$ ($\nu=1,2,\cdots$), $\bigcap_{\nu=1}^{\infty} x_\nu$ exists.[1] R is said to be a normed semi-ordered linear space, if a norm $\|x\| (x \in R)$ is defined and satisfies the condition:

$$|x| \leqq |y| \quad \text{implies} \quad \|x\| \leqq \|y\|$$

in addition to the usual conditions.

A norm $\|x\| (x \in R)$ on a normed semi-ordered linear space is said to be *monotone complete*, if, when $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu \geqq 1} \|x_\nu\| < +\infty$, there exists $\bigcup_{\nu=1}^{\infty} x_\nu$.

A norm on R is said to be *continuous*, if $x_\nu \downarrow_{\nu=1}^{\infty} 0$ implies $\lim_{\nu \to \infty} \|x_\nu\| = 0$ and *semi-continuous*, if $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty} x$ implies $\sup_{\nu \geqq 1} \|x_\nu\| = \|x\|$. It is clear that continuity implies semi-continuity.

Kantorovitch [4] has proved that, if a norm on R is monotone complete and continuous, then it is complete, namely, R is a Banach lattice. Nakano [5; Theorem 31.7] has proved that, if a norm on R is monotone complete and semi-continuous, then the norm is complete, and, recently, Amemiya [1] has proved that, if a norm on R is monotone complete, it is complete.[2] In this connection, see also [2].

In this paper, we will consider several problems concerning monotone completeness and completeness of normed semi-ordered linear spaces and Nakano spaces.

1. Monotone completeness of normed semi-ordered linear spaces. In this section, we will consider two problems.

As usual, let (c_0) be the set of all null-sequences of real numbers. This is a normed semi-ordered linear space by the usual ordering and

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[1] Namely, a conditionally σ-complete vector lattice. In this paper we use the terminology and notation of [5].

[2] In this paper, Amemiya also proved the following lemma: Let R be a monotone complete normed semi-ordered linear space. Then there exists a number $\gamma > 0$ such that $0 \leqq x_\nu \uparrow_{\nu=1}^{\infty} x$ implies $\gamma \|x\| \leqq \sup_{\nu \geqq 1} \|x_\nu\|$.