A DETERMINANT IN CONTINUOUS RINGS

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1. Introduction. In the theory, developed by Dieudonné [1], of determinants of nonsingular square matrices over a noncommutative field K the determinantal values are cosets modulo the commutator subgroup of $K^{\times}$, the multiplicative group of K. Since the matrix groups $M_n^{\times}(K)$ and their commutator subgroups C_n have the property that $M_n^{\times}(K)/C_n$ is independent of n, the latter cosets will serve just as well for determinantal values, at least for theorems involving only the multiplication of determinants.

The rings whose principal right ideal lattices form continuous geometries have many resemblances to matrix rings; in fact, the axioms of Continuous Geometry are satisfied by finite dimensional geometries over a field which are always equivalent to the right ideal lattice of some matrix ring. Irrespective of questions as to the existence or otherwise of fields in connection with a general continuous geometry playing a similar role to that of the field of coordinate values in the finite dimensional case we will show that multiplicative determinantal theorems can be obtained for the more general ring; the determinants will be cosets of the group of invertible ring elements modulo the closure of its commutator subgroup with respect to the rank-distance topology in the ring.

The definition of a complete rank ring is given by von Neumann [3, (iv)]. Essential properties of such a ring $\mathfrak{R}$ and the associated lattice of principal right ideals have been developed by von Neumann [3, 4] and Ehrlich [2]. We will assume throughout that $\mathfrak{R}$ is a complete rank ring, of characteristic not 2; and that if the discrete case (matrices over a field) applies, then the order of the matrices is at least 3.

2. Groups in a complete rank ring. Using a notation similar to that of [2], [3] we denote by $\mathfrak{G}$ the group of invertible ring elements; that is, $u \in \mathfrak{G} \subset \mathfrak{R}$ if and only if the rank $R(u)$ of u is 1.

DEFINITION 1. We denote by $\mathfrak{K}$ the closure of the commutator subgroup of $\mathfrak{G}$ in the rank-distance topology and by $\mathfrak{K}^{\dagger}$ the closure of the group generated by the elements of class 2 in $\mathfrak{G}$.

COROLLARY 1. *$\mathfrak{K}$ and $\mathfrak{K}^{\dagger}$ are groups.*

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