BOUNDS FOR THE PRINCIPAL FREQUENCY OF THE NONHOMOGENEOUS MEMBRANE AND FOR THE GENERALIZED DIRICHLET INTEGRAL

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Introduction. In § § 1 and 2 of this paper we consider an arbitrarily shaped membrane of variable density and uniform tension. We assume that this nonhomogeneous membrane is stretched in a given frame and obtain bounds for its principal frequency (fundamental tone). Before describing our results we quote the analogous result for the nonhomogeneous string proved in a paper by P. R. Beesack and the author [1, Theorem 2].

Let $p(x)$ be continuous and not identically zero for $-x_0 \leqq x \leqq x_0$, $0 < x_0 < \infty$, and let $p^+(x)$ and $p^-(x)$ be the rearrangement of $p(x)$ in symmetrically increasing respectively decreasing order. Consider the three differential systems

$$y''(x)+\lambda p(x)y(x)=0\ , \qquad y(\pm x_0)=0\ ;$$

$$u''(x)+\lambda^+ p^+(x)u(x)=0\ , \qquad u(\pm x_0)=0\ ;$$

$$v''(x)+\lambda^- p^-(x)v(x)=0\ , \qquad v(\pm x_0)=0\ :$$

denote their least positive eigenvalues also by λ, λ^+ and λ^- respectively. Then $\lambda^- \leqq \lambda$ even if $p(x)$ changes sign finitely often while $\lambda \leqq \lambda^+$ holds if $p(x) \geqq 0$.

For the nonhomogeneous membrane we consider a domain D bounded by a Jordan curve C. The differential system (for the original density) is given by

$$\Delta u(x, y)+\lambda p(x, y)u(x, y)=0$$

for (x, y) in D and $u(C)=0$. We base the existence of the first eigenfunction and its minimum property on the classical treatment of Courant-Hilbert [3, vol. 2, Chapter VII]. We assume therefore that $p(x, y)$ is positive and continuous in $\bar{D}$ and has continuous first derivatives in D. Together with $p(x, y)$ we consider its rearrangements in symmetrically increasing respectively decreasing order. The symmetrization is with respect to a point: $p^+(x, y)=p^+(r)$ and $p^-(x, y)=p^-(r)$ are defined in a closed disk $\bar{D}^*$ of the same area as D. The properties of $p(x, y)$ imply that $p^+(x, y)$ and $p^-(x, y)$ are positive and continuous in $\bar{D}^*$. However,

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