ON SEMI-NORMAL OPERATORS

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1. A bounded linear operator A in a Hilbert space will be called semi-normal if

$$H=AA^*-A^*A\geqq 0 \quad (\text{or } \leqq 0)\,. \tag{1}$$

If A is a finite matrix, for instance, then relation (1) implies $H=0$, so that A is even normal; cf., e.g., [4]. That (1) may hold with $H\neq 0$ is seen if one chooses, for instance, A to to the isometric matrix defined by $A=D=(d_{ij})$ where $d_{i+1,i}=1$ and $d_{ij}=0$ otherwise. The purpose of this note is to investigate the spectrum of the semi-normal operator A and of the associated self-adjoint operators J_θ defined by

$$J_\theta=\frac{A_\theta+A_\theta^*}{2}\,, \quad A_\theta=Ae^{-i\theta}\ (\theta \text{ real})\,. \tag{2}$$

It is seen that, in particular, J_θ becomes the real or the imaginary part of A according as $\theta=0$ or $\theta=\pi/2$.

A number λ belonging to the spectrum of A (sp(A)) will be called accessible if there exists a sequence of numbers λ_n not belonging to sp(A) for which $\lambda_n\rightarrow\lambda$ as $n\rightarrow\infty$. If M is any self-adjoint operator, $\max M$ and $\min M$ will denote the greatest and the least points respectively of the set sp(M).

The following theorems will be proved:

THEOREM 1. *Let A be semi-normal with $H\geqq 0$ and let $\lambda=re^{i\theta}$ (r real, $\geqq 0$) be an accessible point of the spectrum of A. Then*

$$(\max J_\theta)^2\geqq \min AA^* \tag{3}$$

and

$$|r-\max J_\theta|\leqq((\max J_\theta)^2-\min AA^*)^{1/2}\,, \tag{4}$$

where J_θ is defined by (2).

THEOREM 2. *Let A be semi-normal and let $J=J_\theta$ have the spectral resolution $J=\int\lambda dE$. Then, if $S=S_\theta$ is any measurable set for which*

$$\int_S dE=I\,, \tag{5}$$

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