ON THE INEQUALITY $\Delta u \geqq f(u)$

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We are interested in solutions of the non-linear differential inequality

$$\Delta u \geqq f(u) \tag{1}$$

where $u(x_1, \cdots, x_n)$ is to be defined in some region of Euclidean n-space and $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian of u. Wittich [5] considered the corresponding equation

$$\Delta u = f(u) \tag{1a}$$

in two dimensions and found conditions on $f(u)$ which guarantee that (1a) has no solution valid in the whole plane. Haviland [1] found a slightly weaker result in 3 dimensions, and Walter [4] generalized Wittich's theorem to n-dimensions. The method is essentially the same in all three papers, resulting on the one hand in the requirement that the function $f(u)$ be convex, and on the other hand in a rather involved argument for the n-dimensional case. The proofs do extend immediately to the inequality (1).

In the present paper we deal directly with (1), and obtain in particular a simple proof of a stronger theorem (Theorem 1 below) where the convexity of $f(u)$ is no longer required. Our method also yields much more precise information on the behavior of solutions.

Recently Redheffer [3] has obtained in the two-dimensional case an improvement of our Theorem 1, where the monotonicity of $f(u)$ is not needed. Although Redheffers's theorem may very likely be extendable to n dimensions, it does not seem possible by his method to obtain the more precise results mentioned in the remarks following Theorem 1.

The present investigation resulted from an attempt to determine the type of a class of Riemann surfaces. One result, Theorem 2, is given here as an application of Theorem 1.

We should like to mention finally that the method presented here has been developed independently by Keller, who, in a paper to be published, derives further information on the behavior of solutions of (1a), and applies his results to an interesting physical problem described in [2].

Notation. Throughout this paper we shall reserve r for the polar

Received March 11, 1957. Work sponsored by Office of Ordnance Research, U. S. Army, Project No. 1323.