ADDITIVE FUNCTIONALS OF A MARKOV PROCESS

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1. Introduction. We are concerned with functionals of the form $L=\int_0^t V[x(\tau)]d\tau$ where $x(t)$ is a temporally homogeneous Markov process in a locally compact Hausdorff space, X, and V is a non-negative measurable function on X. In studying the distribution of this functional various authors (e.g. [1], [3], and [7] have considered the following function

$$r(t, x, A)=E\{e^{-uL}\,|\,x(0)=x; x(t) \in A\}\, p(t, x, A) \tag{1.1}$$

where $p(t, x, A)$ is the transition probability function of $x(t)$. If one can determine r then one can in essence determine the distribution of L since $(u>0)$

$$r(t, x, A)=\int_0^\infty e^{-u\lambda}d_\lambda P[L\leqq \lambda|x(0)=x; x(t) \in A] \qquad p(t, x, A)\ .$$

Formally it is quite easy to see that if p satisfies an equation of diffusion type

$$\frac{\partial p}{\partial t}=\Omega p \tag{1.2}$$

that r should satisfy the equation

$$\frac{\partial r}{\partial t}=(\Omega-uV)r\ . \tag{1.3}$$

If $x(t)$ is the Wiener process in E^N and V satisfies a Lipschitz condition of order $\alpha>0$ Rosenblatt [12] has given a rigorous derivation of (1.3). In this paper we use the theory of semi-groups to give a meaning to (1.3) for a wide class of processes without assuming any smoothness conditions on V. Rosenblatt's result does not follow from ours since our results only imply that r is a "weak" solution of (1.3). However, for many applications (e.g. [10]) this is all that is really required.

Because of certain difficulties connected with the definition of the conditional expectation in (1.1) we define r directly and prove that if $p(t, x, A)>0$ then $\dfrac{r(t, x, A)}{p(t, x, A)}$ is the appropriate conditional expectation. Since we intend to apply analytic methods it is necessary to investigate the dependence of r on its various variables. This is done in § 2.

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