A NOTE ON ADDITIVE FUNCTIONS

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1. A real valued function $f(n)$, defined on the set of natural numbers, is called *additive* if $f(mn)=f(m)+f(n)$ whenever $(m, n)=1$, and *strongly additive* if also $f(p^\alpha)=f(p)$ for p prime and $\alpha=2, 3, \cdots$. We define

$$(1) \qquad A_n=\sum_{p<n} f(p)/p\ , \qquad B_n=\sum_{p<n} f^2(p)/p\ ,$$

and we assume throughout that

$$(2) \qquad B_n \to \infty\ , \qquad n\to\infty\ .$$

Additive functions for which $B_n=O(1)$ have already been discussed thoroughly in Erdös and Wintner [4]. They proved the following theorem:

Define

$$f'(p)=\begin{cases} 1 \text{ for } |f(p)|>1\ , \\ f(p) \text{ for } |f(p)|\leqq 1\ . \end{cases}$$

Then the additive function $f(n)$ *possesses a distribution function if, and only if, the series*

$$\sum_p f'(p)/p \qquad \text{and} \qquad \sum_p \{f'(p)\}^2/p$$

converge.

Moreover, it follows from a general result of P. Lévy [10] that this distribution function is continuous if, and only if, the series $\sum_{f(p)\neq 0} f(p)/p$ diverges. Surveys of this subject are given in Kac [7] and Kubilyus [9]. A comprehensive account is being prepared by H. N. Shapiro.

Our knowledge of functions subject to (2) is not as complete. Outstanding is the result of Erdös and Kac [3] which states that if

$$(3) \qquad f(p)=O(1)\ ,$$

the distribution of

$$\frac{f(m)-A_n}{B_n^{1/2}}\ , \qquad m\leqq n\ ,$$

is asymptotically Gaussian. In a recent note H. N. Shapiro [11] has shown that the theorem of Erdös and Kac remains true even when (3) is replaced by

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