A PROPERTY OF DIFFERENTIAL FORMS IN THE CALCULUS OF VARIATIONS

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1. In the classical problems involving a simple integral

$$I_1 = \int L(t, q^i, \dot{q}^i)\, dt\,, \qquad i=1, \cdots, n\,, \tag{1}$$

one is led to the consideration of the Pfaffian form

$$\omega = L\,dt + \frac{\partial L}{\partial \dot{q}^i}\,\omega^i = \frac{\partial L}{\partial \dot{q}^i}\,dq^i - \left(\dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L\right) dt \tag{2}$$

where

$$\omega^i = dq^i - \dot{q}^i\, dt\,.$$

For example this form ω is the one which gives rise to the "relative integral invariant" of E. Cartan.

In a recent note [1] L. Auslander characterizes the form ω by a theorem equivalent to the following one.

THEOREM 1. *Among all semi-basic forms θ such that*

$$\theta \equiv L\,dt \bmod \omega^i \tag{3}$$

the form ω of (2) *is the only one satisfying the condition*

$$d\theta \equiv 0 \bmod \omega^i\,. \tag{4}$$

In this, a *semi-basic form* is a form for which the local expression contains only the differentials of t, q^i(not of $\dot{q}^i$). The integral I is defined over an arc $\bar{c}$ of a space $\mathscr{W}$ with local coordinates t, q^i, $\dot{q}^i$ satisfying the equations $\omega^i = 0$: Therefore in (1) the form $L\,dt$ may be replaced by any θ satisfying (3).

Condition (4) is a special case of a congruence discovered by Lepage [5]. *The purpose of the present note is to give a natural reason for this congruence which goes beyond its nice algebraic expression.*

Let us observe that the space $\mathscr{W}$ is the manifold of 1-dimensional contact elements of a manifold $\mathscr{V}$ with local coordinates t, q^i. The map

$$(t, q^i, \dot{q}^i) \to (t, q^i)$$

is then the local expression of the natural projection $\pi: \mathscr{W} \to \mathscr{V}$. We

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