NUMERICAL SOLUTION OF VIBRATION PROBLEMS IN TWO SPACE VARIABLES

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1. Introduction. The classical theory of vibrating plates leads to the following non-dimensional fourth order partial differential equation in two space variables $W(x, y, t)$ for the transverse vibrations:

$$\Delta\Delta W + W_{tt} = 0 , \tag{1}$$

where $\Delta\Delta$ is the biharmonic operator

$$\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} .$$

Solutions of this equation for two dimensional regions of arbitrary shape are of course not known, but even for those plate problems for which analytic solutions in series form for this equation are available, the series do not lend themselves easily to numerical calculations. Direct numerical solutions of this equation are therefore of considerable impotance. It is the purpose of this paper to present a new finite difference approximation to this equation which is stable for all values of the mesh ratios $\overline{\Delta t}/\overline{\Delta x}^2$ and $\overline{\Delta t}/\overline{\Delta y}^2$ and which involves an amount of work which is entirely feasible on large-scale digital computers. The method is a generalization of a method prepared by Douglas and Rachford [1] for solving the two dimensional diffusion equation.

2. The differential and difference equations. We consider first the specific problem of determining the transverse vibrations of a square homogeneous thin plate hinged at its boundaries and subjected to an arbitrary initial condition. The boundary value problem may be written

$$(2)\quad \begin{array}{lll}
\text{a)} & \dfrac{\partial^4 W}{\partial x^4} + 2\dfrac{\partial^4 W}{\partial x^2 \partial y^2} + \dfrac{\partial^4 W}{\partial y^4} + \dfrac{\partial^2 W}{\partial t^2} = 0 , & (x, y) \in R , \quad 0 \leqq t \leqq T , \\
\text{b)} & W(x, y, 0) = f(x, y) , & (x, y) \in R , \\
\text{c)} & W_t(x, y, 0) = 0 , & (x, y) \in R , \\
\text{d)} & W(x, y, t) = \dfrac{\partial^2 W}{\partial x^2}(x, y, t) = 0 , & \text{at } x = 0 ,\ 1 \text{ for } 0 < y < 1 ,\ t > 0 , \\
\text{e)} & W(x, y, t) = \dfrac{\partial^2 W}{\partial y^2}(x, y, t) = 0 , & \text{at } y = 0 ,\ 1 \text{ for } 0 < x < 1 ,\ t > 0 ,
\end{array}$$

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