ON SEMI-NORMED *-ALGEBRAS

CHIEN WENJEN

1. Introduction. The notion of semi-normed algebras was introduced by Arens as a generalization of Banach algebras [**2, 5**]. They are called locally multiplically-convex algebras by Michael [**16**]. Various properties of Banach algebras have been generalized to semi-normed algebras [**5, 16, 21, 22, 23**].

We repeat here a few definitions. Let A be a linear algebra over the field K of complex or real numbers. A nonnegative real-valued function V defined on A is called a semi-norm if it satisfies the following conditions:

$V(x+y) \leqq V(x)+V(y)$, $V(xy) \leqq V(x)V(y)$, $V(\lambda x)=|\lambda| V(x)$. Suppose there is a family $\mathscr{V}$ of semi-norms such that $V(x)=0$ for all $V \in \mathscr{V}$ only if $x=0$. A is a semi-normed algebra if all the translations of the sets on which $V(x)<e$, where e is real and $V \in \mathscr{V}$, are taken as a subbase of topology, and is complete if it is complete with respect to the uniform structure defined by the various relations $V(x-y)<e$. A is called an *-algebra if there is a semi-linear operation * such that $(\lambda x-yz)^*=\bar{\lambda}x^* -z^*y^*$, $x^{**}=x$. A subset U of A is called idempotent if $UU \subset U$; it is called multiplicatively convex (m-convex) if it is convex and idempotent. A is locally m-convex if there exists a basis for the neighbourhoods of the origin consisting of sets which are m-convex and symmetric.

The present paper is devoted to generalizing the representation theorems for commutative and noncommutative Banach algebras to semi-normed algebras. An application of the Gelfand-Neumark-Arens representation theorem for commutative Banach algebras yields a simple proof of the spectral theorem for bounded self-adjoint operators in Hilbert space [**14**, p. 95]. Our generalized representation theorem for commutative semi-normed algebras gives rise to a similar proof of the spectral theorem for unbounded self-adjoint operators.

The characterization of the algebra $C(T, K)$ of all complex-valued continuous functions on a locally compact, paracompact Hausdorff space T has been treated by Arens [**5**, p. 469]. We have a characterization theorem for $C(T, K)$ where T is a locally compact completely regular space and also a uniqueness theorem for the space T [cf. the Banach-Stone theorem, **6**, p. 170, 20, p. 469]: If $C(T_1, K)$, $C(T_2, K)$ are topo-

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