STATIONARY MEASURES FOR CERTAIN STOCHASTIC PROCESSES

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Introduction. In a recent paper [1], T.E. Harris has studied stationary processes $\{Z_n\}$ with a finite number of states, taken to be the integers $0, 1, \cdots, D-1$. His technique is to map the half-infinite sample sequences $Z_n, Z_{n-1}, \cdots$ onto the unit interval by means of the correspondence

$$X_{n+1}=Z_n/D+Z_{n-1}/D^2+\cdots . \tag{1}$$

The X_n then form a stationary Markov process. In § 5 of [1] Harris shows (Theorem 7) that if the process $\{Z_n\}$ is of mixing type, then either the stationary distribution $G(x)=\Pr(X_n \leqq x)$ has a unit step, or is the uniform distribution, or $G(x)$ is continuous and totally singular.

The purpose of this paper is to investigate correspondences such as (1) in general, using two simple lemmas in ergodic theory which are given in the next section. If $g(\{i_0, i_1, \cdots\})$ is any essentially one-to-one and measurable mapping of the space of sequences $\{i_0, i_1, \cdots\}$ onto another measurable space X, then a correspondence similar to (1) may be defined between stochastic processes with states i and processes on X:

$$X_{n+1}=g(\{Z_n, Z_{n-1}, \cdots\}) . \tag{2}$$

Theorem 1 describes the resulting distributions on X; Theorem 2 is a specialization to the case of (1). Finally an additional application (Theorem 3) is made to certain of the processes studied by Karlin in [3]. Theorem 2 contains Theorem 7 of [1], and Theorem 3 overlaps with § 7 of [3]. In addition to a unified approach, some extension of the previous results is obtained in both cases.

2. Ergodic theory lemmas.

LEMMA 1. *Let (Ω, W) be a measurable space and T a measurable transformation of Ω onto itself. Let μ_1 and μ_2 be two sigma-finite measures on (Ω, W) such that for each, T is a measure preserving, metrically-transitive transformation. Then if μ_1 and μ_2 are not proportional, they are orthogonal (i.e., have their positive mass on disjoint sets).*

Proof. Suppose μ_1 and μ_2 are both finite measures, and assume they have been normalized. Let A be a set such that $\mu_1(A) \neq \mu_2(A)$. Define

$$B_i=\left\{\omega \in \Omega \,|\, \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n}\phi_A(T^j\omega)=\mu_i(A)\right\} , \qquad i=1, 2,$$

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