CURVATURE IN HILBERT GEOMETRIES

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For every pair of points, p and q, interior to a simple, closed, convex curve C in the Euclidean plane, the line $\xi = p \times q$ cuts C in a pair of points u and v. If C has at most one segment then the Hilbert distance from p to q, defined by

$$h(p, q) = \left| \log\left(\frac{up}{uq} \cdot \frac{vq}{vp} \right) \right| ,$$

is a proper metric (where up denotes the Euclidean distance from u to p), and is invariant under projective transformations. The geometry induced on the interior of C is a Hilbert geometry, and the Hilbert lines are carried by Euclidean lines [2].

We shall be concerned here with curvature at a point defined in a qualitative rather than a quantitative sense (cf. [1, p 237]).

DEFINITION 1. The *curvature at* p is *positive* or *negative* if there exists a neighborhood U of p such that for every x, y in U we have

$$2\ h(\bar{x}, \bar{y}) \geqq h(x, y) ,$$

respectively

$$2\ h(\bar{x}, \bar{y}) \leqq h(x, y) ,$$

where $\bar{x}$, $\bar{y}$ are the Hilbert midpoints respectively of the segments from p to x and p to y. If there is neither positive nor negative curvature at a point then the curvature is *indeterminate* at that point. This qualitative curvature is clearly a projective invariant.

In order to state our result we need one more concept.

DEFINITION 2. A point p is a *projective center* of C if there exists a projective transformation, π, of the plane so that πp is the affine center of πC.

A projective center is characterized by the following. Let ξ be a line through p, and let $\xi \cap C = \{u, v\}$, and let p'_ξ be the harmonic conjugate of p with respect to u and v. Finally, let L_p be the locus of all p'_ξ. Then p is a projective center if and only if L_p is a straight line.

Conic sections are characterized by the fact that every point in their interior is a projective center [3]. We can now state our main result, which solves a problem of H. Busemann [1, Problem 34, p. 406].

THEOREM. *If p is a point of determinate curvature then it is*