A NUMERICAL CONDITION FOR MODULARITY OF A LATTICE

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1. Introduction. In this note a simple numerical condition (θ) is presented which is necessary for modularity of a finite lattice L. Though not sufficient (θ) appears to be a condition imposing a strong tendency toward modularity.

NOTATION. Covering, proper inclusion, and inclusion will be denoted by $>$, $\supset$, $\supseteqq$ respectively. $N[S]$ will denote the order of the set S. The unit and zero elements will be denoted by u and z respectively.

DEFINITION 1. A finite lattice L is upper semi-modular [1: p. 100] if and only if

(ξ') $\quad a$ and $b > a \cap b$ imply $a \cup b > a$ and b.

L is lower semi-modular if and only if

(ξ'') $\quad a \cup b > a$ and b imply a and $b > a \cap b$.

DEFINITION 2. In a finite lattice let $C(a) = \{x \in L | x < x \cup a > a\}$ and $D(a) = \{x \in L | x > x \cap a < a\}$.

2. Tests for modularity An immediate consequence of Definitions 1 and 2 is the following theorem.

THEOREM 1. *In a finite lattice L condition (ξ') is equivalent to $D(a) \subseteqq C(a)$ for all $a \in L$ and both imply $N[D(a)] \leqq N[C(a)]$. Dually, (ξ'') is equivalent to $D(a) \supseteqq C(a)$ for all $a \in L$ and both imply $N[D(a)] \geqq N[C(a)]$. Moreover, modularity, (ξ') and (ξ''), is equivalent to $D(a) = C(a)$ for all $a \in L$ and both imply the condition (θ):*

(θ) $\quad N[D(a)] = N[C(a)]$ for all $a \in L$.

The contrapositive of the last statement of Theorem 1 serves as a useful test for non-modularity:

THEOREM 2. *If there exists $a \in L$ for which $N[D(a)] \neq N[U(a)]$, then L is non-modular.*

When either (ξ') or (ξ'') is known to hold in L, the verification of the condition (θ) is a test often easiest to apply. It merely requires counting coverings.

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