UNIFORM CONTINUITY OF CONTINUOUS FUNCTIONS OF METRIC SPACES

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In this paper we intend to find equivalent conditions under which continuous functions of a metric space are always uniformly continuous. Isiwata has attempted to prove a theorem in a recently published paper [3] by a method that has a close relation with ours. Unfortunately he does not accomplish his purpose, so we shall give a correct theorem (Theorem 3) in the last part of this paper and, for this purpose, give a condition for the existence of a uniformly continuous unbounded function in a metric space (Theorem 2).

In this paper the space S, unless otherwise specified, is the metric space with a distance function $d(x, y)$, and, for a positive number α, the α-sphere about a subset A $\{x ; d(A, x)<\alpha\}$ is denoted by $S(A, \alpha)$; the function is the real valued continuous mapping.

DEFINITION 1. Let us consider a family of neighborhoods U_n of x_n such that $\{x_n\}$ is a sequence of distinct points and $U_m \cap U_n=\phi$ (=empty) for $m\neq n$. Let $f_n(x)$ be a function such that $f_n(x_n)=n$ and $f_n(x)=0$ for $x \notin U_n$. Then a *mapping constructed from the family* is a mapping $f(x)$ defined by $f(x)=f_n(x)$ for x belonging to some U_n and $f(x)=0$ for the other x.

LEMMA. *Consider a family of neighborhoods U_n of x_n satisfying the following conditions*:

(1) *$\{x_n\}$, which consists of distinct points, has no accumulation point,*

(2) *$\overline{U}_m \cap \overline{U}_n=\phi$, $m\neq n$ ($\overline{U}$ a closure of U), and $U_n \subset S(x_n, 1/n)$,*

(3) *there is a sequence of points y_n such that distances of x_n and y_n converge to* 0 *and y_n does not belong to any U_m; then the mapping constructed from the family is continuous and not uniformly continuous. When $\{x_n\}$ is a sequence containing infinitely many distinct points and has no accumulation point, there is a family of neighborhoods of x_n satisfying* (2); *if $\{x_n\}$ further contains infinitely many distinct accumulation points, then the family besides satisfies* (3).

Proof. The continuity of the mapping constructed from the family follows from $\overline{\cup U_{n_i}} = \cup \overline{U}_{n_i}$ for any subsequence $\{n_i\}$ of indices; the mapping is not uniformly continuous by (3). Suppose $\{x_n\}$ consists of distinct accumulation points and has no accumulation point, then, by an inductive process, we have neighborhood V_n of x_n such that $V_n \subset S(x_n, 1/n)$ and $\overline{V}_m \cap \overline{V}_n=\phi$, and have y_n and a neighborhood U_n of x_n

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