ON THE PRINCIPAL FREQUENCY OF A MEMBRANE

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1. **Let D denote a simply-connected region** in the xy-plane whose boundary consists of a finite number of piecewise smooth arcs. If λ is the principal frequency of a homogeneous membrane which covers D and is kept fixed at its boundary C, then, according to a well-known theorem of Rayleigh [3], λ is not smaller than the principal frequency of a circular membrane of equal area and density. This may also be expressed by saying that the homogeneous circular membrane has the lowest principal frequency among all homogeneous membranes of the same mass.

In this paper we shall be concerned with the possible generalizations of Rayleigh's theorem to the case of non-homogeneous membranes. It is clear that no general result of this type is to be expected unless certain restrictions are imposed on the density distribution of the membrane. Indeed, it is easily shown that the principal frequency of a membrane of given mass can be made arbitrarily small if enough of the mass is concentrated in a small area interior to D. It is therefore necessary to add conditions which prevent the excessive accumulation of mass at interior points of the membrane. As the following theorem shows, a sufficient condition of this type is the requirement that the density distribution $p(x, y)$ be such that $\log p(x, y)$ is subharmonic, i.e., that the mean value of $\log p(x, y)$ on any circular circumference inside D is not smaller than the value of $\log p(x, y)$ at the center.

THEOREM I. *If λ is the principal frequency of a membrane of given mass whose density distribution $p(x, y)$ is such that $\log p(x, y)$ is subharmonic, then*

$$\lambda \geqq \lambda_0 , \tag{1}$$

where λ_0 is the principal frequency of a homogeneous circular membrane of the same mass.

The conclusion of Theorem I will in general not hold if the restriction on $p(x, y)$ is replaced by the somewhat weaker condition that $p(x, y)$ be subharmonic. The following theorem shows, moreover, that — at least in the case of a circular membrane — inequality (1) is reversed if $p(x, y)$ is assumed to be superharmonic.

THEOREM II. *If λ is the principal frequency of a circular membrane*

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