THE REPRESENTATION OF AN ANALYTIC FUNCTION BY GENERAL LAGUERRE SERIES

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1. Introduction. Hille [4] has solved the problem of finding necessary and sufficient conditions that a function be represented by Hermitian series in a strip. Pollard [7] has solved the analogous problem in a strip for Laguerre series of order zero. We propose to solve the problem for Laguerre series of order $\alpha(\alpha > -1)$ getting as a region of convergence a parabola instead of a strip. From this theorem the generalization of Pollard's result follows immediately.

We say that a function of a complex variable $f(z)$ where $z = x + iy = re^{i\theta}$ possesses a *Laguerre series of order* $\alpha(\alpha > -1)$ *or a general Laguerre series if*

$$f(z) \sim \sum_{n=0}^{\infty} a_n^{(\alpha)} L_n^{(\alpha)}(z) \qquad (n = 0, 1, 2, \cdots) \tag{1.1}$$

where

$$a_n^{(\alpha)} = \left\{\binom{n+\alpha}{n}\Gamma(\alpha+1)\right\}^{-1}\int_0^{\infty} e^{-x}x^{\alpha}L_n^{(\alpha)}(x)f(x)\,dx \qquad (\alpha > -1) \tag{1.2}$$

$L_n^{(\alpha)}(x)$ is the Laguerre polynomial of order $\alpha > -1$ and degree n given by [8 p. 97 formula 5.1.6] and the above series converges. The series is said to be the *Laguerre expansion* of $f(z)$.

We define

$$d_\alpha \equiv -\limsup_{n} (2n^{\frac{1}{2}})^{-1} \log |a_n^{\alpha}| \tag{1.3}$$

and by the notation

$$z \in p(b) \quad b > 0\,; \qquad z \in \overline{p}(b) \tag{1.4}$$

we mean respectively that z lies in the open (closed) parabolic region

$$p(b) : y^2 < 4b^2(x + b^2)\,; \qquad \overline{p}(b) : y^2 \leqq 4b^2(x + b^2)\,.$$

If we select that branch of $z^{\frac{1}{2}}$ for which $(-z)^{\frac{1}{2}}$ is real and positive when $z < 0$ then $\Re(-z)^{\frac{1}{2}} = \{\frac{1}{2}(r - x)\}^{\frac{1}{2}} = b^2$ gives the equation $y^2 = 4b^2(x + b^2)$ of the parabola which is the boundary of the above regions.

The main result of this paper is the following.

THEOREM A. *In order that the function $f(z)$ possess a Laguerre series of order* α $(\alpha > -1)$ (or a general Laguerre series) *which converges to it*

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