ON THE LEBESGUE AREA OF A DOUBLED MAP

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If X is a metric space and A is a non-empty closed subset of X we construct a space Y by doubling X about A in such a way that X is imbedded homeomorphically in Y, the image of A is the boundary of the image of X, and X is also homeomorphic to the closure of the complement of its homeomorphic image in Y. In this way any function f on X may be doubled in a natural way to yield a function F on Y. In 17 it is shown that if X and A satisfy certain triangulability conditions, and f is continuous to Euclidean n space, E_n, with $n \geqq k \geqq 2$, then $L_k(F) \leqq 2L_k(f)$, with L_k denoting k-dimensional Lebesgue area. In 18, 21 and 22 the restrictions of 2-dimensionality are used to show that, when $k = 2$, we have in fact $L_2(F) = 2L_2(f)$.

In particular if (X, A) is a 2-dimensional manifold with boundary, then Y is a compact 2-dimensional manifold. Furthermore, if X is finitely triangulable, then X and A satisfy the required triangulability conditions and $L_2(F) = 2L_2(f)$. Thus to compute the Lebesgue area of f, we need only to know the Lebesgue area of F, whose domain is a compact 2-dimensional manifold.

Our terminology is consistent with [1]; however, some additional notations are cited below

1. NOTATIONS.

(i) 0 is the empty set,

(ii) $\{x\}$ is the set whose sole element is x.

(iii) $\sigma A = \{x \mid \text{for some } y, x \in y \in A\}$.

(iv) R is the set of real numbers.

(v) $A^\cap = \{x \mid x \subset A\}$.

(vi) $N(f, A, y)$ is the number of elements, possibly infinite, in the set $\{x \mid x \in A \text{ and } y = f(x)\}$.

(vii) $\operatorname{dmn} f = \{x \mid \text{for some } y, (x, y) \in f\}$.

(viii) $\operatorname{rng} f = \{y \mid \text{for some } x, (x, y) \in f\}$.

2. AGREEMENT.

(i) If X is a topological space and i is a positive integer, then $X^i = \{A \mid A \text{ is an } i\text{-cell in } A\}$.

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