TWO THEOREMS OF GAUSS

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The purpose of this note is to show that two famous theorems of Gauss can be derived from a common source. The theorems alluded to are the following:

THEOREM 1. (The triangular-exponent identity)

$$\prod_{s=1}^{\infty} \frac{1-x^{2s}}{1-x^{2s-1}} = \sum_{s=1}^{\infty} x^{s(s-1)/2} . \tag{1}$$

THEOREM 2. (The evaluation of Gauss sums)

$$\sum_{s=0}^{m-1} e^{2\pi i s^2/m} = \begin{cases} \sqrt{m} & \textit{for} \quad m \equiv 1 \pmod 4 \\ i\sqrt{m} & \textit{for} \quad m \equiv 3 \pmod 4 . \end{cases} \tag{2}$$

Both these results will be obtained as consequences of the following identity previously stated by the author [2] without proof.

A finite identity.

THEOREM. *If* $P_0 = 1$ *and*

$$P_n = \prod_{s=1}^{n} \left(\frac{1-x^{2s}}{1-x^{2s-1}} \right)$$

for $n = 1, 2, \cdots$, *then*

$$A_n = \sum_{s=0}^{n-1} \frac{P_n}{P_s} x^{s(2n+1)} = \sum_{s=1}^{2n} x^{s(s-1)/2} = S_n \tag{3}$$

and

$$A'_n = \sum_{s=1}^{n} \frac{P_n}{P_s} x^{s(2n+1)} = \sum_{s=1}^{2n+1} x^{s(s-1)/2} = S'_n . \tag{3'}$$

Proof. We readily verify that

$$(1-x^{2n})x^{s(2n+1)} = (1-x^{2n-1})x^{s(2n-1)} + (1-x^{2s+1})x^{(s+1)(2n-1)} - (1-x^{2s})x^{s(2n-1)} ,$$

and by multiplying by $\dfrac{P_{n-1}}{P_s(1-x^{2n-1})}$ we find

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