THE FREE LATTICE GENERATED BY A SET OF CHAINS

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1. Introduction. P. M. Whitman [4] defined an ordering of the set of lattice polynomials generated by a set of unrelated elements. R. P. Dilworth [3] generalized this ordering to apply to the case of lattice polynomials generated by an arbitrary partly ordered set P. Dilworth proved that this ordering gives a lattice isomorphic to the free lattice, FL(P), which is generated by P and which preserves bounds of pairs of elements of P. R. A. Dean [2] considered the ordering of lattice polynomials which preserves order of pairs of elements in P and which leads to the completely free lattice CF(P). He shows that CF(P) and FL(P) are identical in the case in which P is a set of unrelated chains.

This article is a further study of FL(P) where P is a set of unrelated chains. An arbitrary element of P will be denoted by p or q. The set of chains consisting of

$$a_{11}<a_{12}< \cdots <a_{1n_1}\ ;\ a_{21}<a_{22}< \cdots <a_{2n_2}\ ;\ \cdots\ ;\ a_{m1}<a_{m2}< \cdots <a_{mn_m}\ ;$$

where a_{ij} and a_{kl} are unrelated when $i \neq k$, will be denoted by $n_1+n_2+\cdots+n_m$.

DEFINITION 1. *Lattice polynomials* over P are defined inductively as follows.

(1) The elements $p, q, \cdots$, of P are lattice polynomials over P.

(2) If A and B are lattice polynomials over P, then so are $A \cup B$ and $A \cap B$.

DEFINITION 2. The *rank*, $r(A)$, of a lattice polynomial A is defined inductively as follows.

(1) $r(A)=0$ if and only if A is in P.

(2) $r(A \cup B)=r(A \cap B)=r(A)+r(B)+1$.

DEFINITION 3. The dual polynomial, A', of a polynomial A of FL$(n_1+n_2+\cdots+n_m)$ is defined inductively as follows.

(1) If $A \equiv a_{ij}$, then $A' \equiv a_i(n_i-j+1)$.

(2) If $A \equiv A_1 \cup A_2$, then $A' \equiv A_1' \cap A_2'$.

(3) If $A \equiv A_1 \cap A_2$, then $A' \equiv A_1' \cup A_2'$.

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