A PRÜFER TRASFORMATION FOR DIFFERENTIAL SYSTEMS

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1. Introduction. For a real self-adjoint matrix differential equation

$$(1.1) \qquad (P(x)Y')' + F(x)Y = 0 \,, \qquad a \leqq x < \infty ,$$

in the $n \times n$ matrix $Y(x)$ it has been established recently by J. H. Barrett [1] that there is a transformation analogous to the well-known Prüfer [8] polar-coordinate transformation for a real self-adjoint linear homogeneous differential equation of the second order. In the form for a solution $Y(x)$ of (1.1) obtained by Barrett the roles of the sine and cosine functions in the Prüfer transformation are assumed by the respective $n \times n$ matrices $S(x)$, $C(x)$ satisfying a matrix differential system

$$(1.2) \qquad S = Q(x)C \,, \quad C' = -Q(x)S \,, \quad S(a) = 0 \,, \quad C(a) = E \,,$$

where $Q(x)$ is an associated real symmetric matrix. Barrett uses the method of successive approximations to determine $Q(x)$ as a solution of the functional equation $Q = CP^{-1}C^* + SFS^*$, where S and C are related to Q by (1.2).

The present paper is concerned with the derivation of similar results for a matrix differential system

$$(1.3) \qquad Y' = G(x)Z \,, \quad Z' = -F(x)Y$$

where $G(x)$, $F(x)$ are continuous $n \times n$ hermitian matrices; in particular, if $G(x)$ is of constant rank and $G(x) \geqq 0$ then (1.3) is equivalent to a differential system with complex coefficients that is of the general form of the canonical accessory differential equations for a variational problem of Bolza type. The method of the present paper for the determination of the associated matrix $Q(x)$ is more direct than that employed by Barrett [1]; in particular, the present method affords a ready determination of the most general form of $Q(x)$. In addition, it is shown that certain criteria of oscillation and non-oscillation obtained by Barrett for an equation (1.1) may be improved and extended.

Matrix notation is used throughout; in particular, matrices of one column are termed vectors, and for a vector (y_α), $(\alpha = 1, \cdots, n)$, the norm $|y|$ is given by $(|y_1|^2 + \cdots + |y_n|^2)^{\frac{1}{2}}$. The symbol E is used for

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