CONTINUOUS PRODUCTS AND NONLINEAR INTEGRAL EQUATIONS

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1. Introduction. J. S. MacNerney [2] has expressed the solution M of the Stieltjes integral equation

$$M(x, t) = 1 + \int_x^t dF(u) \cdot M(u, t)$$

as a continuous product

$$M(x, t) = {}_x\Pi^t [1 + dF]$$

in case F is a function from the real numbers into the space B of continuous linear transformations from a normed, linear and complete space S into S, which is continuous and of bounded variation on each interval. This is a generalization of the familiar relationship

$$e^{t-x} = 1 + \int_x^t e^{t-u} du, \quad e^{t-x} = \lim_{n\to\infty}\left[1 + \frac{t-x}{n}\right]^n .$$

The object of this paper is to extend these considerations to a larger class of integral equations including nonlinear equations. The space S is required to be an additive abelian group with zero element N, having a norm $\|\cdot\|$ such that, if x and y are in S then $\|x\| > 0$ unless $x = N$, and $\|-x\| = \|x\|$, $\|x + y\| \leqq \|x\| + \|y\|$. The space S is complete with respect to the metric induced by this norm. The function F from a number interval into the set B of all continuous transformations from S into S is required to satisfy certain inequalities (Theorem A′, § 3).

In § 2 we develop a continuous product in a still more general setting, requiring only that S should be a complete metric space, and then specialize in § 3. Having in mind the problem of numerical solution of differential and integral equations, we give particular attention to obtaining upper bounds to the errors in various approximations to the continuous product (§ § 3 and 4).

In § 5, integral equations of the form

$$Y(t) = A + \int_c^t dF \cdot Y$$

are solved by means of the continuous product. Section 6 contains

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