INJECTIVE MODULES OVER NOETHERIAN RINGS

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Introduction In this discussion every module over a ring R will be understood to be a left R-module. R will always have a unit, and every module will be unitary. The aim of this paper is to study the structure and properties of injective modules, particularly over Noetherian rings. B. Eckmann and A. Schopf have shown that if M is a module over any ring, then there exists a unique, minimal, injective module $E(M)$ containing it. The module $E(M)$ will be a major tool in our investigations, and we shall systematically exploit its properties.

In § 1 we show that if a module M has a maximal, injective submodule C (as is the case for left-Noetherian rings), then C contains a carbon-copy of every injective submodule of M, and M/C has no injective submodules different from 0. Although C is unique up to an automorphism of M, C does not in general contain every injective submodule of M. In fact, the sum of two injective submodules of a module is always injective if and only if the ring is left-hereditary.

In § 2 we show that for any ring R a module E is an indecomposable, injective module if and only if $E \cong E(R/J)$, where J is an irreducible, left ideal of R. We prove that if R is a left-Noetherian ring, then every injective R-module has a decomposition as a direct sum of indecomposable, injective submodules. Strong uniqueness assertions can be made concerning such decompositions over any ring.

In § 3 we take R to be a commutative, Noetherian ring, and P to be a prime ideal of R. We prove there is a one-to-one correspondence between the prime ideals of R and the indecomposable, injective R-modules given by $P \leftrightarrow E(R/P)$. We examine the structure of the module $E = E(R/P)$, and show that if A_i is the annihilator in E of P^i, then $E = \cup A_i$ and A_{i+1}/A_i is a finite dimensional vector space over the quotient field of R/P. The ring of R-endomorphisms of E is isomorphic in a natural way to $\overline{R}_p$, the completion of the ring of quotients of R with respect to R-P. As an $\overline{R}_p$-module E is an injective envelope of $\overline{R}_p/\overline{P}$, where $\overline{P}$ is the maximal ideal of $\overline{R}_p$. If P is a maximal ideal of R, then E is a countably generated R-module. Every indecomposable, injective R-module is finitely generated if and only if R has the minimum condition on ideals.

In § 4 we take R to be a commutative, Noetherian, complete, local ring, P the maximal ideal of R and $E = E(R/P)$. Then the contravariant,

Received April 10, 1958. In revised form June 9, 1958.