MULTIPLICITY FREE REPRESENTATIONS OF FINITE GROUPS

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Introduction. Wigner in [5] has defined a finite group $\mathscr{G}$ to be simply reducible if (a) every conjugate class in $\mathscr{G}$ is self inverse and (b) for each two irreducible representations L and M of $\mathscr{G}$ the Kronecker product $L \otimes M$ is a direct sum of inequivalent irreducible representations. The principal result of [5] is a curious purely group theoretical characterization of simply reducible groups. For each $x \in \mathscr{G}$ let $v(x)$ denote the number of elements of $\mathscr{G}$ which commute with x and let $\zeta(x)$ denote the number of solutions of the equation $y^2 = x$. Then $\mathscr{G}$ is simply reducible if and only if $\sum_{x \in G} v(x)^2 = \sum_{x \in G} \zeta(x)^3$. As the author has shown in [3] this result may be "explained" as follows. Let $\tilde{\mathscr{G}}_3$ be the diagonal subgroup of $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$, that is the set of all x, y, z with $x = y = z$. Then it is easily seen that the number of $\tilde{\mathscr{G}}_3 : \tilde{\mathscr{G}}_3$ double cosets in $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$ is equal to $\sum_{x \in G} v(x)^2$ while the number of self inverse $\tilde{\mathscr{G}}_3 : \tilde{\mathscr{G}}_3$ double cosets in $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$ is equal to $\sum_{x \in G} \zeta(x)^3$. Thus Wigner's condition is equivalent to the condition that every $\tilde{\mathscr{G}}_3 : \tilde{\mathscr{G}}_3$ double coset be self inverse. On the other hand if H is an arbitrary subgroup of the finite group $\mathscr{G}$ and U^I is the corresponding permutation representation of $\mathscr{G}$ one can prove that every $H : H$ double coset is self inverse if and only if each irreducible component M_j of U^I occurs with multiplicity one and is such that the intertwining operators of M_j with $\bar{M}_j$ are symmetric. This result is a corollary of a general theorem on anti-symmetric intertwining numbers for induced representations and certain elementary lemmas. It leads easily to Wigner's theorem when applied to $\mathscr{G} \times \mathscr{G} \times \mathscr{G}$ and its diagonal subgroup.

Now of the two conditions in the definition of simple reducibility (b) is much the more interesting. Moreover, as we shall see, there are examples of groups which satisfy (b) and not (a). This suggests looking for a generalization of Wigner's theorem in which (a) is dropped or weakened. The way to such a generalization is suggested by the considerations of [3] and a simple observation which plays a vital role in Gelfand's work [1] on "spherical functions" on Lie groups. Slightly generalized[1] and then applied to finite groups this observation is the following. Let $x \to x^a$ be an involutory anti-automorphism of the finite group $\mathscr{G}$. Let H be a subgroup of $\mathscr{G}$ such that the $H : H$ double cosets are invariant under $x \to x^a$. Let $\mathscr{A}_H$ be the subalgebra of the

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[1] Compare Mautner [4].