SOME EXTENSIONS OF A THEOREM OF MARCINKIEWICZ

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1. Introduction. Let $F(x)$ be a distribution function that is a nondecreasing, right continuous function such that $F(-\infty) = 0$ and $F(+\infty) = 1$. The Fourier transform of $F(x)$, that is, the function

$$f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x) \tag{1.1}$$

is called the characteristic function of $F(x)$. It is often of interest to decide whether a given function $f(t)$ can be a characteristic function, i.e., whether it admits the representation (1.1). Necessary and sufficient conditions are known which a complex-valued function of a real variable t must satisfy in order to be a characteristic function (see e. g. [7]). However, these general conditions are not easily applicable. Therefore various conditions were derived which are restricted to certain classes of functions but are applied more readily.

J. Marcinkiewicz [10] derived necessary conditions for an entire function to be a characteristic function. In the course of this study he obtained the following result:

THEOREM A. *An entire function of finite order $\rho > 2$ whose exponent of convergence ρ_1 is less than ρ can not be a characteristic function.*

As a consequence he obtained also

THEOREM B. *Let $P_m(t)$ be a polynomial of degree $m > 2$ and denote by $f(t) = \exp[P_m(t)]$. Then $f(t)$ can not be a characteristic function.*

Theorem B is frequently called Marcinkiewicz' theorem. This theorem is quite often useful and was applied by many authors in studies concerning the statistical characterization of the normal distribution. A short while before the publication of Marcinkiewicz' paper G. Kunetz proved [5], [6] certain particular cases of the theorem. He did not however succeed in proving the theorem for arbitrary polynomials. Marcinkiewicz based his proof on the classical theory of entire functions. More recently D. Dugué [3] gave a new proof of Theorem B and showed that the result was due to certain convexity properties of characteristic

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